Problem 1. (20 points – 5 points for each part)

(a) [11.1: Sequences] Give the mathematical definition of what is meant by: \( \lim_{n \to \infty} a_n = L \).

(b) [11.2: Series] Give the mathematical definition of what is meant by: \( \sum_{n=1}^{\infty} a_n = S \).

(c) [11.2: Series] If it is known that \( \sum_{n=1}^{\infty} a_n = S \), what can be said about \( \lim_{n \to \infty} a_n \)?

(d) [11.10: Taylor and Maclaurin Series] Given an analytic function \( f(x) \), write down its general Taylor series centered at \( a \).

Problem 2. (20 points) [11.1: Sequences] Determine whether the following sequence converges or diverges. If it converges, find the limit.

\[ a_n = 2 + \frac{(-1)^n + 8^n + 2}{9^n}, \quad n = 1, 2, 3, \ldots \]
Problem 3. (20 points) [11.4-11.6: Comparison Test, Alternating Series, Absolute Convergence] Is the following series absolutely convergent, conditionally convergent, or divergent? (Justify your answer.) If it converges, estimate $R_3 = S - S_3$, which is the remainder error when using the 3rd partial sum $S_3$ as an approximation to the sum of the series $S$.

$$
\sum_{n=1}^{\infty} \frac{2(-1)^n}{3+n}
$$

Problem 4. (20 points) [11.8,11.10: Power Series, Taylor and Maclaurin Series] Starting with the general form of a Taylor series, derive the MacLaurin series for $f(x) = 2e^{3x}$, and show that it converges for all $x$. Furthermore, show that $f(x)$ is actually analytic, by showing it is actually the sum of its MacLaurin series. (Hint: To show that $f(x)$ is analytic, use Taylor’s inequality and the Squeeze Theorem to show $\lim_{n \to \infty} R_n(x) = 0$, where $f(x) = T_n(x) + R_n(x)$, with $T_n(x)$ the $n$-th degree Taylor polynomial and $R_n(x)$ the remainder.)

Problem 5. (20 points) [1.1-1.2: Simple Linear ODE Models of Growth and Decay] Find the general solution of the following ODE, then solve the associated IVP. (Hint: After finding the general solution, use the initial condition to determine the integration constant appearing in the general solution.)

$$2y' + 4ty = 0, \quad y(0) = 3.$$ 

Finally, evaluate the solution at $t = 3$. 

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