2.5

5. The data are spaced 90 years apart. We can express the solution of the logistic equation $y' = ky\left(1 - \frac{y}{M}\right)$ in the form $y = \frac{M}{1+Av^{t/90}}$, where A is a constant and $v = e^{-90k}$. Substituting $y(0) = 4$, $y(90) = 50$, and $y(180) = 203$ we have $4(1 + A) = M$, $50(1 + Av) = M$, and $203(1 + Av^2) = M$. Solve these equations: $4(1 + A) = 50(1 + Av)$, so $A = \frac{23}{2-25v}$. Substitute this expression for A in $203(1+Av^2) = 4(1+A)$ and simplify to obtain the quadratic equation $4669v^2 - 4975v + 306 = 0$, or $(4669v - 306)(v-1) = 0$. Hence $v = 306/4669 \approx 0.06554$ or $v = 1$ (the second root is extraneous). Then $A = 23/(2 - 25v) = 107387/1688 \approx 63.6$, and $M = 4 + 4A = 109075/422 \approx 258.5$. Thus $y = \frac{258.5}{1+63.6(0.065539)^{t/90}}$.

6. The predicted carrying capacity is $\lim_{t\to\infty} y(t) = 258.5$.

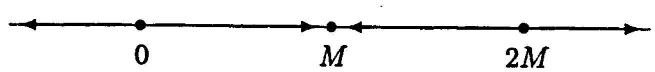
7. Solve $\frac{258.5}{1+63.6(0.065539)^{t/90}} = 210$ to obtain

$$\frac{1}{63.6}\left(\frac{258.5}{210} - 1\right) = 0.065539^{t/90}$$

or $0.0036313 = 0.065539^{t/90}$. Taking logarithms, we have $\frac{t}{90}(-2.7351) = -5.6182$ It follows that $t = 185.5$, and therefore $T = 1790 + 185.5 = 1975.5$.

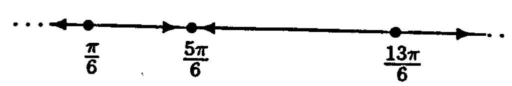
8. $y(2050 - 1790) = y(260) = \frac{258.5}{1+63.6(0.97017)^{260}} = 252.4$ million.

10. $f(y) = k(M - y)(2M - y)y$, Let $f(y) = 0$, we obtain 3 stationary values: $y_1 = 0$, $y_2 = M$, and $y_3 = 2M$. The function $f(y)$ is positive when $0 < y < M$ and when $y > 2M$; and f is negative for $M < y < 2M$. Here is the phase diagram:



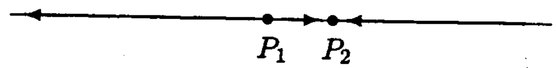
You can see that $y = 0$ and $y = 2M$ are unstable stationary points, and $y = M$ is stable. If the initial population is between 0 and $2M$, the solution will converge to the stable population M. If the population initially exceeds $2M$ then the population will go to infinity, which does not make sense.

20. Solving $f(y) = \sin(y) - \frac{1}{2} = 0$ we get two stationary points $y = 2k\pi + \frac{\pi}{6}, 2k\pi + \frac{5\pi}{6}$ in each interval $[2k\pi, 2(k+1)\pi]$, where k is an integer. Because $f'(y) = \cos y$, we have $f'(2k\pi+\frac{\pi}{6}) = \frac{\sqrt{3}}{2} > 0$, and $f'(2k\pi+\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2} < 0$, the stationary point $y = 2k\pi + \frac{\pi}{6}$ is unstable and $y = 2k\pi + \frac{5\pi}{6}$ is stable.

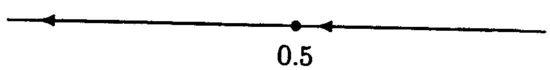


29.

(a) The critical harvest rate is the maximum harvest rate that the population can sustain without becoming extinct. Let $f(P) = P(1-P) - H = -P^2 + P - H$ be the growth rate. The stationary points are the roots of $f(P) = 0$. By the quadratic formula, for $H < 0.25$ there are two stationary points, $P_1, P_2 = \frac{1 \pm \sqrt{1-4H}}{2}$. The phase diagram for this case is



As the harvest rate increases, the stationary point P_1 moves to the right, and P_2 moves left, until $H = 0.25$, when there is only one stationary point, $P = 0.5$, and the phase diagram looks like this:

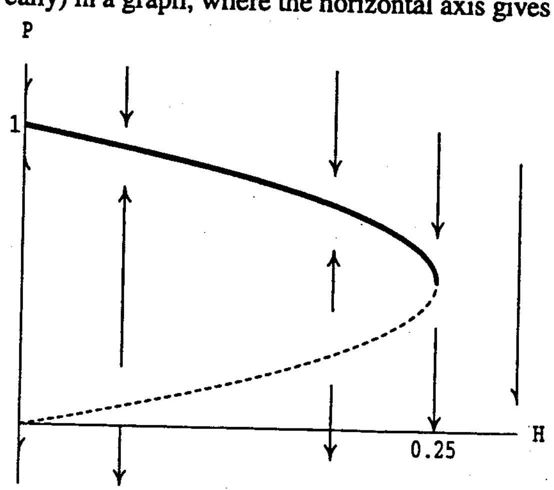


When $H > 0.25$ there is no stationary point, and the population becomes extinct. Thus the critical harvest rate is 0.25.

Notice that 0 is not a stationary point. The model predicts that populations will become negative if $P(0) < P_1$ (in the case $H < 0.25$), and for $P(0) < 0.5$ in the critical case. If the harvest rate exceeds this critical value, negative populations are predicted for any starting population. Of course the model will cease to apply when the resource is exhausted; the model actually tell us that that the state of extinction will not be approached asymptotically, it will occur in a finite span of time.

(b) For $H = 0.1$ we have $P_1, P_2 = \frac{1 \pm \sqrt{1-0.6}}{2} = 0.5 \pm 0.316$. $P_1 = 0.184$, $P_2 = 0.816$. Referring to the first phase diagram in part (a), (P_1, P_2) is an up interval, and $(-\infty, P_1)$ and $(P_2, +\infty)$ are down intervals. Therefore $y = P_2$ is stable. Because $P(0) = 1 > P_2$, the limiting population will be $P_2 = 0.816$

(c) The bifurcation diagram merges the phase diagrams (presented vertically) in a graph, where the horizontal axis gives the harvest rate.



3.1

5. Substituting $x = e^{2t}(t+1)$ and $y = e^{2t}(t-1)$ in the first equation yields the identity $2e^{2t}(t+1) + e^{2t} = e^{2t}(t+1) + e^{2t}(t-1) + 3e^{2t}$. The same substitution in the second equation results in $2e^{2t}(t-1) + e^{2t} = 2e^{2t}(t+1) - 3e^{2t}$.

Substituting $x = C\,e^{2t}$ and $y = C\,e^{2t}$ in the first equation of the "associated homogeneous system" yields the identity $2C\,e^{2t} = C\,e^{2t} + C\,e^{2t}$. The same substitution in the second equation results in $2C\,e^{2t} = 2C\,e^{2t}$.

A family of solutions of the inhomogeneous system is the sum of the associated homogeneous solutions plus a particular solution. Thus we obtain a family of solutions: $x = e^{2t}(t+1+C)$, $y = e^{2t}(t-1+C)$

7. The ODE can be written as $\frac{d^2y}{dt^2} = -3\frac{dy}{dt} - 4y + t^2$ Set $v = \frac{dy}{dt}$ and $v' = \frac{d^2y}{dt^2}$. Then the system

$$\begin{aligned} y' &= v \\ v' &= -3v - 4y + t^2 \end{aligned}$$

replaces the given ODE.

8. The ODE can be written as $\frac{d^2z}{dt^2} = (1 - z^2)\frac{dz}{dt} - z$ Set $v = \frac{dz}{dt}$ and $v' = \frac{d^2z}{dt^2}$. Then the system

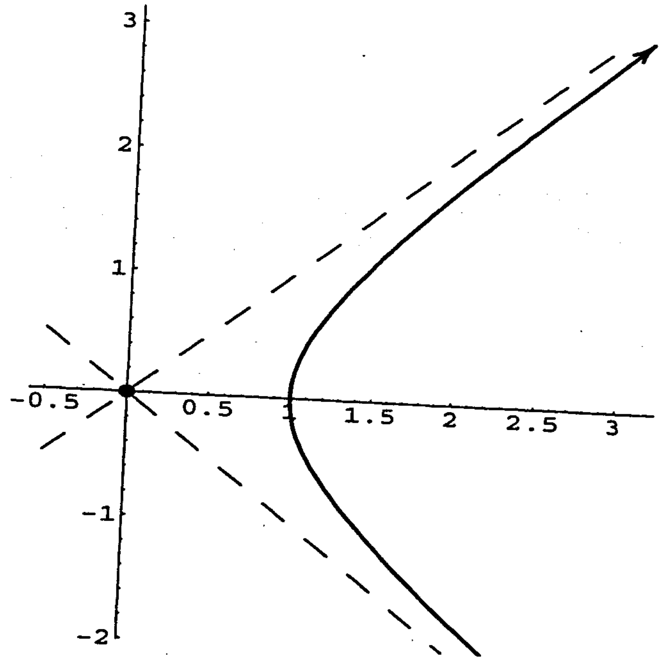
$$\begin{aligned} z' &= v \\ v' &= (1 - z^2)v - z \end{aligned}$$

replaces the given ODE.

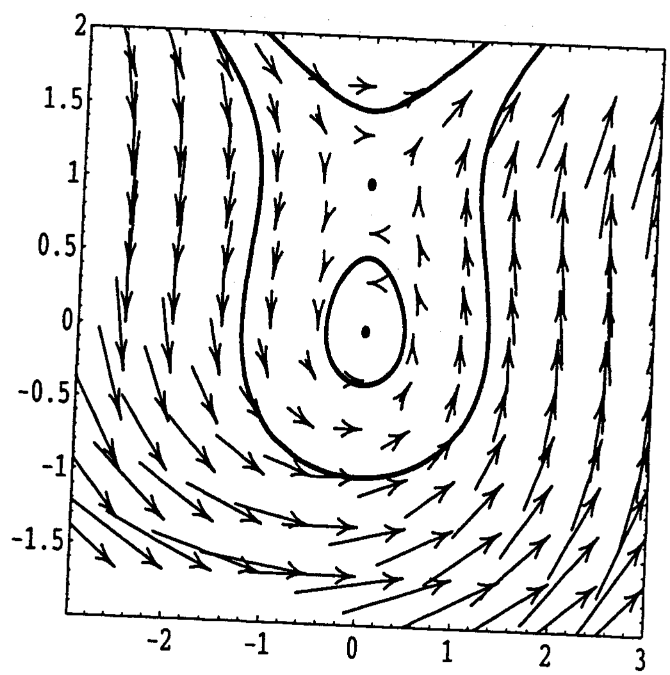
3.2

6. Substituting $x = \sec(t)$, and $y = \tan(t)$ in the first equation yields the identity $\sec(t)\tan(t) = \sec(t)\tan(t)$. The same substitution in the second equation results in $\sec^2(t) = \sec^2(t)$.

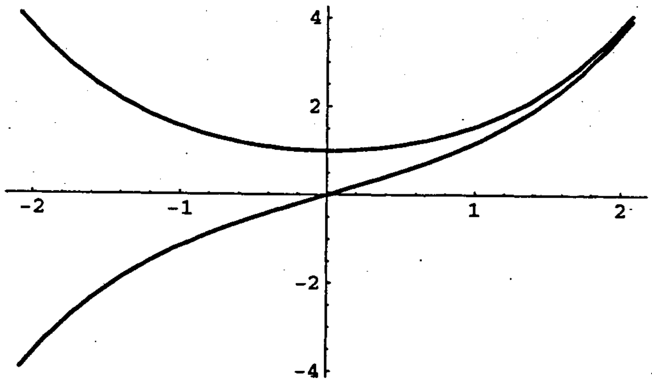
The identity $\sec^2(t) = \tan^2(t) + 1$ can be used to eliminate t in the solution $x = \sec(t)$, $y = \tan(t)$. We have $y^2 = x^2 + 1$, and as $t \to \pi/2$, both x and $y \to \infty$. This orbit is therefore the segment of the hyperbola $y = \sqrt{x^2+1}$ in the first quadrant and fourth quadrants, directed upward.



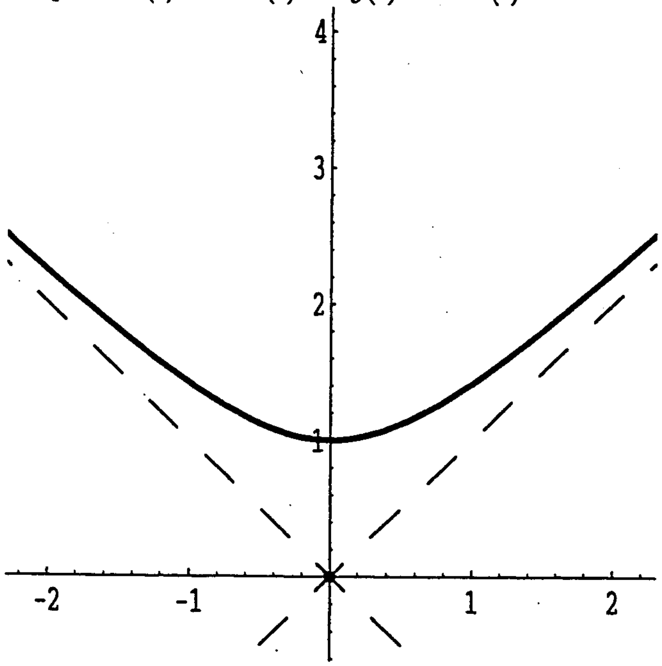
10. By solving the equations $(y^2 - y)/4 = 0$, $x/4 = 0$ we get the stationary points $(0,0)$, and $(0,1)$.



#14



Graphs of $x(t) = \sinh(t)$ and $y(t) = \cosh(t)$.



Phase Portrait