1.3.1. Find the general solution of the ODE \( y' = 3t - 4y \) on the interval \((-\infty, \infty)\). Then find the particular solution that satisfies the initial condition \( y(0) = 0 \).

Rewriting as \( y' + 4y = 3t \), we multiply by the factor \( e^{\int 4dt} = e^{4t} \) to get \( e^{4t}y' + 4e^{4t}y = 3te^{4t} \). This integrating factor should allow simplification of the left side of the equation, and indeed it becomes \( \frac{d}{dt}e^{4t}y \). Integrating both sides with respect to \( t \) yields that \( e^{4t}y = \int 3te^{4t}dt \). Integration by parts of the right side yields \( \frac{3}{4}t e^{4t} - \frac{3}{16} e^{4t} + C \), so \( y = \frac{3}{4}t - \frac{3}{16} + Ce^{-4t} \), which is valid over the entire interval. Plugging in \( t = 0 \) and \( y = 0 \) yields \( 0 = 0 - \frac{3}{16} + C \), so \( C = \frac{3}{16} \) and the solution to the initial value problem is \( y = \frac{3}{4}t + \frac{3}{16}(e^{-4t} - 1) \).

1.3.5. Find the general solution of the ODE \( y' + 2ty = e^{-t^2} \) on the interval \((-\infty, \infty)\). Then find the particular solution that satisfies the initial condition \( y(0) = 0 \).

We multiply by the factor \( e^{\int 2tdt} = e^{t^2} \) to get that \( e^{t^2}y' + 2te^{t^2}y = 1 \). The left side of this equation simplifies to \( \frac{d}{dt}e^{t^2}y \), so integrating both sides with respect to \( t \) gives \( e^{t^2}y = t + C \), so \( y = te^{-t^2} + Ce^{-t^2} \), which is valid over the entire interval. Plugging in \( t = 0 \) and \( y = 0 \) yields \( 0 = 0 + C \), so \( C = 0 \) and the solution to the initial value problem is \( y = te^{-t^2} \).

1.3.13. A penny is heated to 800°C and is then allowed to cool. The temperature after a minute is 600°C, and the room temperature is 20°C. When will it be safe to pocket the coin (the temperature should be less than 50°C)?

By Newton’s law of cooling, the temperature \( y \) is governed by the differential equation \( y' = -k(y - 20) \) for some constant \( k \). Rearranging this yields \( y' + ky = 20k \), and multiplication by the factor \( e^{\int kdt} = e^{kt} \) gives us \( \frac{d}{dt}e^{kt}y = 20ke^{kt} \), which upon integration with respect to \( t \) gives \( e^{kt}y = \int 20ke^{kt}dt = 20e^{kt} + C \). Thus \( y = 20 + Ce^{-kt} \), and it remains only to determine the values of \( C \) and \( k \) to completely determine the behavior of \( y \). We know that \( y(0) = 800 \) and \( y(1) = 600 \), so \( 800 = 20 + C \) and \( 600 = 20 + Ce^{-k} \); the former gives us \( C = 780 \) and the latter \( k = -\ln \frac{580}{780} \approx 0.296 \). Now to solve the question actually asked, we must find a \( t \) such that \( y(t) = 50 \), that is, \( 50 = 20 + 780e^{-0.296t} \), which when solved gives \( t = \frac{-1}{0.296}\ln \frac{30}{580} \approx 11 \), so after 11 minutes it would be safe to touch the penny.

1.3.24. Find the periodic solution of each of the following ODEs and show that it is stable.

(c) \( y' + 5y = \cos(\pi t) \).

We multiply by the integrating factor \( e^{\int 5dt} = e^{5t} \) to get \( e^{5t}y' + 5e^{5t}y = e^{5t}\cos(\pi t) \). The left side of this is equal to \( \frac{d}{dt}e^{5t}y \), so integrating both sides with respect to \( t \) gives \( e^{5t}y = \int e^{5t}\cos(\pi t) \). Using the formula \( \int e^{at}\cos(bt) \) and integrating by parts, we get \( y = \frac{5}{\pi^2 + 25}(e^{5t}\cos(\pi t) + \pi e^{-5t}) \), which is valid over the entire interval. Then \( y = \frac{5}{\pi^2 + 25}(e^{5t}\cos(\pi t) + \pi e^{-5t}) \), so the long term \( y \) is approximately equal to \( \frac{5}{\pi^2 + 25} \) regardless of starting value, since the exponential term approaches zero as \( t \) grows.

(f) \( y' + py = \cos(\omega t) \), where \( p > 0 \) is a constant. What would happen if \( p < 0 \)?
We multiply by the integrating factor $e^{\int p \, dt} = e^{q t}$ to get $e^{q t} y' + pe^{q t} y = e^{q t} \cos(\omega t)$. The left side of this is equal to $\frac{d}{dt} e^{q t} y$, so integrating both sides with respect to $t$ gives $e^{q t} y = \int e^{q t} \cos(\omega t) = \frac{e^{q t} (pcos(\omega t) + \omega sin(\omega t)) + C}{\omega^2 + p^2}$, as determined by a computer-algebra system (alternatively, a table of integrals or integration by parts can be used). Then $y = \frac{pcos(\omega t) + \omega sin(\omega t)}{\omega^2 + p^2} + C e^{-q t}$, so over the long term $y$ is approximately equal to $\frac{pcos(\omega t) + \omega sin(\omega t)}{\omega^2 + p^2}$ regardless of starting value, since the exponential term approaches zero as $t$ grows. This would not be the case for $p < 0$, however, in which the exponential term is either constant (in the case $p = 0$) or grows without bound; In the latter case the periodic solution is unstable, since solutions tend to vary from it by increasing amounts as $t$ grows, and in the $t = 0$ case there are an infinitude of periodic solutions, which are neither stable nor unstable.

2.1.5. An object in free fall is subject to gravitational force of $mg$ and frictional force of $-bv$. Show that the terminal velocity $v_\infty$ satisfies the equation $mg - bv_\infty = 0$, and therefore $v_\infty = \frac{mg}{b}$.

The differential equation expressing the velocity of the object is $v' = mg - bv$. The limiting velocity is a velocity at which $v' = 0$, since such a condition mandates a stable value which under normal conditions solution curves will not cross. Thus the terminal velocity is given by $0 = mg - bv_\infty$.

2.1.8. Find a family of solutions for the ODE $y' = 3t^2 y^2$. Then find the particular solution that satisfies the initial condition $y(0) = -1$ and specify its domain.

We rearrange this equation into the separated form $\frac{dy}{y^2} = 3t^2 \, dt$, which upon evaluation of the implicit integrals yields $\frac{1}{y} = t^3 + C$, so $y = \frac{1}{t^3 + C}$ is a solution on either of the domains $(-\infty, -\sqrt[3]{C})$ or $(-\sqrt[3]{C}, \infty)$ (since the solution is undefined when $t^3 + C = 0$). Given the initial condition $y(0) = -1$, we evaluate the general form to obtain $-1 = \frac{1}{-C}$ so that $C = 1$, so our equation is $y = \frac{1}{t^3 + 1}$ on either the domain $(-\infty, -1)$ or $(-1, \infty)$.

Since we were given a value of $y$ for $t = 0$, we must be considering the latter domain.

2.1.12. Find a family of solutions for the ODE $dy = (y^2 + 1) \, dt$. Then find the particular solution that satisfies the initial condition $y(0) = 1$ and specify its domain.

We rearrange this equation into the separated form $\frac{dy}{y^2 + 1} = dt$, which upon evaluation of the implicit integrals yields $\tan^{-1} y = t + C$, so $y = \tan(t + C)$ is a solution on any domain where $y$ is defined; in particular, the boundaries of the domain are the points where $t + C = \pi n + \frac{\pi}{2}$. Given the initial condition $y(0) = 1$, we evaluate the general form to obtain $1 = \tan(0 + C)$ so that $C = \frac{\pi}{4}$, so our equation is $y = \tan(t + \frac{\pi}{4})$ on some domain which does not include values $t = \pi n + \frac{\pi}{2}$. In particular, to include $t = 0$, which is our initial condition, our maximal domain is $\left(\frac{-3\pi}{4}, \frac{\pi}{4}\right)$.

2.1.16. If an object is initially moving upward (we consider upward velocity to be negative), then the force of air resistance will be downward (positive), and the ODE

$$v' = g + \frac{k}{m} v^2$$

Thursday, October 21, 2004 Page 2 of 6
will apply as long as \( v \leq 0 \), but when \( v > 0 \), the sign of the quadratic term should be changed to minus.

(a) Show that the ODE

\[
v' = g - \frac{k}{m}v|v|
\]

is applicable to both upward and downward motion.

We note that \( v|v| = v^2 \) when \( v > 0 \) and \( v|v| = -v^2 \) when \( v < 0 \) (and all three are of course identical for \( v = 0 \)), so this ODE satisfies the description given.

(b) An object is thrown upward with an initial velocity of 100 meters/second \( v(0) = -100 \) subject to the differential equation

\[
v' = 9.8 + 9.8 \times 10^{-4}v^2.
\]

Give its velocity as a function of time for the duration of its upward motion.

We rearrange this equation into the separated form \( \frac{dv}{9.8 + 9.8 \times 10^{-4}v^2} = dt \), so integrating both sides and rearranging gives \( \tan^{-1} \left( \frac{v}{100} \right) = 9.8 \times 10^{-2}t + C \). Plugging in the values \( t = 0 \) and \( v = -100 \) yields \( \tan^{-1}(-1) = C \), so \( C = \frac{\pi}{4} \), and the equation becomes

\[
v = 100 \tan \left( 9.8 \times 10^{-2}t - \frac{\pi}{4} \right).
\]

(c) Adapt the solution of Example 2.1.4 to obtain the velocity as a function of time for downward motion.

From the previous part of this exercise, we know that the velocity is zero when \( 9.8 \times 10^{-2}t - \frac{\pi}{4} = 0 \), so we reach zero velocity at time \( t = \frac{\pi}{39.2 \times 10^{-2}} \). Example 2.1.4 gives an equation for an object’s fall from rest starting at time \( t = 0 \), so all that need be done is to “recenter” the equation given so that \( t = \frac{\pi}{39.2 \times 10^{-2}} \) corresponds to \( t = 0 \) in the original example:

\[
v = 100 \frac{e^{0.0196(\frac{\pi}{39.2 \times 10^{-2}}) - 1}}{e^{0.0196(\frac{\pi}{39.2 \times 10^{-2}}) + 1}}.
\]

2.3.18. (a) Show that the solution of the IVP

\[
y' = f(x); y(a) = b,
\]

is

\[
y(x) = b + \int_a^x f(u)du.
\]

We rearrange this equation into the separated form \( dy = f(x)dx \), so \( y = \int_a^x f(u)du + C \). Our lower bound here is arbitrary, since any change in it can be incorporated into our added constant \( C \), so we might as well make \( y = \int_a^x f(u)du + C \). Applying the initial condition \( y(a) = b \), we find that \( b = \int_a^a f(u)du + C = C \). Thus our solution is \( y(x) = b + \int_a^x f(u)du \).
(b) Show that approximating the solution to the aforementioned IVP by Euler’s method amounts to calculating a Riemann sum for the integral in question. Euler’s method gives us that \( y(a + kh) \approx y(a + (k - 1)h) + hf(a + (k - 1)h) = y(a + (k - 1)h) + hf(a + (k - 1)h) \). Arguing inductively will show that, if we derive each \( y(a + kh) \) in this manner, then \( y(a + kh) \approx y(a) + \sum_{n=1}^{k} hf(a + (n - 1)h) \). The former addend is simply \( b \), and the latter is clearly the left-hand Riemann sum with step \( h \) of the integral \( \int_{a}^{a+kh} f(u)du \).

2.3.21. Let \( y = \phi(t) \) denote the solution of the IVP,

\[
y' = t + y; y(0) = 0.
\]

Since the differential equation is linear, you can solve this IVP and find a formula for \( \phi(t) \). This problem asks you to calculate \( \phi(1) \) and to approximate \( \phi(1) \) by using Euler’s method. The purpose is to see how fast the approximation improves when the time step is reduced.

Use Euler’s method with time steps \( h = 1, 0.5, 0.25, 0.1, 0.05, 0.02, \text{ and } 0.01 \) to approximate \( \phi(1) \), and let \( E(h) = |\phi(1) - y_N| \) (where \( Nh = 1 \)) denote the approximation error obtained with time step \( h \). Plot a graph of \( E(h) \) as a function of \( h \).

The solution of this differential equation follows a familiar pattern: we rearrange the differential equation as \( y' - y = t \) and multiply by the integrating factor \( e^{\int -t dt} = e^{-t} \) to yield \( e^{-t}y' - e^{-t}y = e^{-t}t \). The left side is simply \( \frac{d}{dt}e^{-t}y \), so integrating both sides with respect to \( t \) results in the equation \( e^{-t}y = C - te^{-t} - e^{-t} \), or \( y = Ce^t - t - 1 \). Plugging in the initial conditions will yield that \( C = 1 \), so \( \phi(t) = e^t - t - 1 \), and in particular \( \phi(1) = e - 2 \approx 0.71828 \).

Below a table indicates the steps in estimating \( y(1) \) using Euler’s method with step \( h = 0.25 \). Similar procedures are followed for other values of \( h \), and may be determined by hand or (in the case of very small \( h \)) with a computer or calculator.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( y(t) )</th>
<th>( y' = y + t )</th>
<th>( \Delta y = hy' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.25</td>
<td>0</td>
<td>0.25</td>
<td>0.0625</td>
</tr>
<tr>
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<td>0.0625</td>
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<td>0.140625</td>
</tr>
<tr>
<td>0.75</td>
<td>0.203124</td>
<td>0.953125</td>
<td>0.23828125</td>
</tr>
<tr>
<td>1.00</td>
<td>0.44140625</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The calculation being used to redetermine \( y \) at each line is \( y(x + h) = y(x) + \Delta y(x) \). Applying this for several values of \( h \) we get the following estimations of \( \phi(1) \):

<table>
<thead>
<tr>
<th>( h )</th>
<th>( N )</th>
<th>( y_N )</th>
<th>( E(h) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0.718281</td>
</tr>
<tr>
<td>0.50</td>
<td>2</td>
<td>0.25</td>
<td>0.468282</td>
</tr>
<tr>
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<td>4</td>
<td>0.441406</td>
<td>0.276878</td>
</tr>
<tr>
<td>0.1</td>
<td>10</td>
<td>0.593742</td>
<td>0.124539</td>
</tr>
<tr>
<td>0.05</td>
<td>20</td>
<td>0.653298</td>
<td>0.064984</td>
</tr>
<tr>
<td>0.02</td>
<td>50</td>
<td>0.691588</td>
<td>0.026693</td>
</tr>
<tr>
<td>0.01</td>
<td>100</td>
<td>0.704814</td>
<td>0.013468</td>
</tr>
</tbody>
</table>
2.3.25. The solution $y = e^{-100t}$ of the initial value problem

$$y' = -100y; y(0) = 1$$

converges to 0 very rapidly. This is apparent even if we restrict our attention to $0 \leq t \leq 1$, since $e^{-100 \times 1}$ is of the order of magnitude of $10^{-43}$.

(a) Use Euler's method with $h = 0.1$ to approximate the solution of the initial value problem for $0 \leq t \leq 1$. Does your computed solution appear to converge to 0?

Using Euler's method, the apparent value of $y$ diverges rapidly and alternates sign; $y(0.1)$ is computed to be $-9$, $y(0.2)$ is computed to be $81$, and so forth, until $y(1)$ is computed to be approximately $3.5 \times 10^9$.

(b) How small should $h$ be to ensure that the solution converges to 0?

To converge to zero, we want the absolute values of the terms to diminish. Note that Euler's method assumes that $y(t + h) = y(t) + hy'(t)$, which in this particular case is $y(t) - 100h y(t) = (1 - 100h)y(t)$. Thus the absolute value of $y$ is diminishing if $|1 - 100h| < 1$, so $1 - 100h$ must be greater than $-1$, giving that $h$ must be greater than $\frac{1}{10} = 0.1$.

2.3.26. One approach that avoids erratic behavior observed in the solution computed in Exercise 2.3.25 is to use backward differences instead of forward differences with Euler's method. Thus we use the difference equation

$$\nabla y_{m+1} = hf(t_{m+1}, y_{m+1})$$

for $m \geq 0$,

where $\nabla y_{m+1} = y_{m+1} - y_m$, as a model for the differential equation $y' = f(t, y)$. This difference scheme is said to be implicit because it defines $y_{m+1}$ implicitly by means of the relation

$$y_{m+1} - y_m - hf(t_{m+1}, y_{m+1}) = 0.$$

Test the backward Euler method by approximating the solution of the initial value problem in Exercise 2.3.25, again with $h = 0.1$, for $0 \leq t \leq 1$.

First we must solve the implicit relation given: $y_{m+1} - y_m - h(-100y_{m+1}) = 0$, which will give that $(1 + 100h)y_{m+1} = y_m$, so $y_{m+1} = \frac{y_m}{1 + 100h}$, which inductively extends to tell us that $y_m = \frac{y_0}{1 + 100h^m}$, so in the case $h = 0.1$, finding $y(1) = y_{10}$ gives us a very small number indeed (approximately $3.9 \times 10^{-11}$).

2.4.2. Solve the IVP $y' - 5y = 25t; y(0) = 10$ and find the largest interval on which the solution is defined.
By the existence theorem, this differential equation’s solution should be valid over all values of \( t \). We multiply by the factor \( e^{5t} \) to get \( e^{-5t}y' - 5e^{-5t}y = 25e^{-5t}t \); the left side simplifies to \( \frac{d}{dt} e^{-5t}y \), so integration of both sides with respect to \( t \) gives \( e^{-5t}y = -5te^{-5t} - e^{-5t} + C \) (with the value on the right obtained via integration by parts). The general solution is thus \( y = -5t + C e^{5t} \), and evaluating at the initial value \( y(0) = 10 \) gives \( 10 = -1 + C \), so \( C = 11 \) and the solution, valid at all values of \( t \), is \( y = 9e^{5t} - 5t - 1 \).

2.4.22. Determine which of the following functions satisfy a Lipschitz condition with respect to \( y \) on the domain

\[
D = \{(t, y) : -1 < t < 1, -1 < y < 1\},
\]

and find a Lipschitz constant for those that do.

(a) \( f(t, y) = t - y^2 \).

Note that \( |(t - y_2^2) - (t - y_1^2)| = |y_1^2 - y_2^2| = |y_1 - y_2| \cdot |y_1 + y_2| \), so our Lipschitz constant (if such exists) would be given by the maximum value of \( |y_1 + y_2| \), which is 2. Since we have just shown that \( |f(t, y_2) - f(t, y_1)| \leq 2|y_2 - y_1| \), the Lipschitz condition is satisfied.

(b) \( f(t, y) = 4ty \).

Note that \( |4ty_2 - 4ty_1| = 4|t| \cdot |y_2 - y_1| \), so our Lipschitz constant (if such exists) would be given by the maximum value of \( 4|t| \), which is 4. Since we have just shown that \( |f(t, y_2) - f(t, y_1)| \leq 4|y_2 - y_1| \), the Lipschitz condition is satisfied.

(c) \( f(t, y) = |y| \).

When \( y_1 \) and \( y_2 \) have the same sign, then \( |y_1 - y_2| = |y_1 - y_2| \), and when \( y_1 \) and \( y_2 \) have opposite signs, \( |y_1 - y_2| = |y_1 + y_2| \leq |y_1 - y_2| \), so in each case \( |f(t, y_2) - f(t, y_1)| \leq |y_2 - y_1| \), so the Lipschitz condition is satisfied with constant 1.

(d) \( f(t, y) = |y| \).

Note that \( |1| = 1 \), and that \( |1 - \epsilon| = 0 \) for small positive \( \epsilon \). Thus, taking \( y_2 = 1 \) and \( y_1 = 1 - \epsilon \), we find that \( |f(t, y_2) - f(t, y_1)| = 1 \), while \( |y_2 - y_1| = \epsilon \), so it is necessary, if the Lipschitz condition is satisfied, that \( K > \frac{1}{\epsilon} \). However, since \( \epsilon \) can be as small as we like, this would force \( K \) to be arbitrarily large, so the Lipschitz condition is unsatisfied.

(e) \( f(t, y) = |t| \).

Since \( f(t, y) \) does not depend on \( y \), it follows that \( f(t, y_2) - f(t, y_1) = 0 \), so \( |f(t, y_2) - f(t, y_1)| \leq 0|y_2 - y_1| \), satisfying the Lipschitz condition with constant 0.