Problem 1	
Problem 2	
Problem 3	
Problem 4	
TOTAL:	

Name and TA Section:	
Student Number:	

SAMPLE Midterm #2, Math 20D, 2004 Fall Quarter

Place/Time: PCYNH/MULTI 106, 9:00-9:50am, 19 November 2004

Instructions: Solve the following four problems. If you need extra paper, first use the back of each page of the exam. If necessary, additional paper can be provided and should be stapled to the exam (in this case, clearly indicate which problem you are solving). (NOTE: Anything in red or blue would not appear on a real exam.)

Problem 1. (25 points) [1.3,2.2: Linear first order ODE, exact equations, integrating factors]
We are given the following IVP:

$$\begin{aligned} ty' + y &= 1, \text{ on } (1, \infty), \\ y(1) &= 2. \end{aligned}$$

(a) (20 points) Find the general solution.

SOLUTION: This is a linear first-order inhomogenous ODE of the form: $a_1(t)y' + a_0(t)y = g(t)$, where $a_1(t) = t$, $a_0(t) = 1$, $g(t) = 1$. We can use either an integrating factor, or find the homogeneous solution and then use variation of parameters. In either case, we will first write it in the form:

$$y' + p(t)y = f(t), \quad \text{where} \quad p(t) = \frac{a_0(t)}{a_1(t)} = \frac{1}{t}, \quad f(t) = \frac{g(t)}{a_1(t)} = \frac{1}{t}.$$

Integrating factor approach: We first compute the integrating factor

$$m(t) = e^{\int p(t)dt} = e^{\int \frac{1}{t}dt} = e^{\ln t} = t.$$

We then have $[ty]' = [my]' = my' + mp(t)y = mf(t) = t(\frac{1}{t}) = 1$, so that $y(t) = Ct^{-1} + 1$.

Homogeneous solution plus variation of parameters approach: We first solve the homogeneous problem: $y_h' + p(t)y_h = 0$, which is

$$y_h(t) = Ce^{\int -p(t)dt} = Ce^{\int -\frac{1}{t}dt} = Ce^{\ln t^{-1}} = Ct^{-1}.$$

(As usual, we have $y_h(t) = 1/m(t)$, ignoring the constant.) We then find a particular solution in the form $y_p(t) = y_h(t)w(t)$. We know that $w(t)$ always satisfies the ODE:

$$y_h w' = f,$$

which comes from simply plugging $y_p(t)$ into the ODE $y' + p(t)y = f(t)$, and using the fact that $y_h(t)$ is the homogeneous solution. (Note if we work with $a_1(t)y' + a_0(t)y = g(t)$, then the ODE $w(t)$ satisfies is just slightly different: $a_1 y_h w' = g$; we get exactly the same result for $w(t)$.) Now, $\frac{1}{t}w'(t) = y_h(t)w'(t) = f(t) = \frac{1}{t}$, giving $w' = 1$, or $w(t) = t$. Finally then $y_p = y_h w = t^{-1}t = 1$, and so $y(t) = y_h(t) + y_p(t) = Ct^{-1} + 1$.

(b) (5 points) Find the specific solution corresponding to the given initial condition.

SOLUTION: We have that: $2 = y(1) = C(1)^{-1} + 1 = C + 1$, so that $C = 1$, and thus the solution is $y(t) = t^{-1} + 1$.

Problem 2. (25 points) [2.1,2.3-2.5: Separable ODE, graphical analysis, nonlinear growth models]
A well-known model for the nonlinear dynamics of a species is the *logistic growth model*:

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right),$$

where $r > 0$ and $K > 0$.

(a) (5 points) Determine the stationary solutions.

SOLUTION: This autonomous ODE has the form: $y' = f(y)$. We can find stationary solutions of autonomous ODEs by simply finding y such that $f(y) = 0$ (since this implies $y' = 0$). Clearly, since $f(y)$ is a degree 2 polynomial in y, it has two roots, which by inspection are $y_1 = 0$ and $y_2 = K$.

(b) (5 points) Determine the stability of the stationary solutions.

SOLUTION: Stability is determined by $g(y) = \partial f(y)/\partial y$, evaluated at $y = y_1$ and $y = y_2$. Computing this, $g(y) = r(1 - \frac{2y}{K})$. Checking each stationary solution, we find $g(y_1) = g(0) = r > 0$, which is unstable, and $g(y_2) = g(K) = -r < 0$, which is stable.

(c) (15 points) Draw the approximate solution plot, showing the asymptotic behavior of solutions that start with non-negative initial data.

SOLUTION: We just need to show what happens to any solution in the intervals $0 = y_1 \le y_2 = K \le +\infty$. Besides what we have already determined about location and stability of the stationary solutions, the only other information we need is given by the ODE itself $y' = f(y)$. I.e., for any $y \in (0, +\infty)$, $f(y)$ tells us when $y(t)$ is increasing or decreasing. (Look in the book for the approximate solution plot for this particular example.)

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Problem 3. (25 points) [(3.1-3.2,3.4-3.5) Systems of ODE, phase plane, interacting populations]
Find an integral $F(x, y) = C$ for the following system of ODEs:

$$x' = x(1-y),$$
$$y' = y(x-1).$$

SOLUTION: We just solve the single ODE for the orbits to produce $F(x, y)$. We will have trouble computing an integrating factor $m(x, y)$ for the orbit ODE, since it will end up involving both variables x and y. However, it turns out to be separable, and as we learned in Problem 22 in Section 2.2, separation of variables can be viewed as the special case where you can actually determine $m(x, y)$.

The ODE for the orbits has the form:

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)} = \frac{y(x-1)}{x(1-y)} = \left(\frac{x-1}{x}\right)\left(\frac{y}{1-y}\right),$$

where we have regrouped things in the last term to set up the separation of variables. Separating, we have:

$$\left(\frac{1-y}{y}\right) dy = \left(\frac{x-1}{x}\right) dx.$$

We now seek $F(x, y)$ such that:

$$dF(x,y) = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = P(x,y)dx + Q(x,y)dy = 0, \quad (1)$$

where in this case, $P(x, y) = P(x) = (1-x)/x$, and $Q(x, y) = Q(y) = (1-y)/y$. Being separable, it is always exact since $\partial P/\partial y = 0 = \partial Q/\partial x$. Since $P(x, y) = P(x)$ and $Q(x, y) = Q(y)$, we know any $F(x, y)$ satisfying (1) must satisfy both $F(x, y) = \int P(x)dx + v(y)$ and $F(x, y) = \int Q(y)dy + u(x)$; one can verify this by computing $\partial F/\partial x$ and $\partial F/\partial y$. This requires $u(x) = \int P(x)dx$ and $v(y) = \int Q(y)dy$, so $F(x, y) = \int P(x)dx + \int Q(y)dy$. Computing these simple integrals gives $F(x, y) = \ln x - x + \ln y - y$, which gives then finally:

$$F(x, y) = \ln x + \ln y - x - y = C.$$

Problem 4. (25 points) [(4.1-4.2,4,4) IVP for linear nonhomogenous ODE systems]
We are given the following ODE system: $v' = Av + f$, where

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad f = \begin{pmatrix} e^t \\ -e^t \end{pmatrix}, \quad v(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad v(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

(a) (10 points) Find the fundamental solution matrix $\mathcal{X}(t)$ and homogeneous solution $v_h = \mathcal{X}(t)c$.

SOLUTION: We look for the fundamental solutions $v_1(t) = e^{r_1 t}b_1$, $v_2(t) = e^{r_2 t}b_2$, where (r_k, b_k), $k = 1, 2$ are the two eigenpairs of A. We first determine the eigenvalues of A by solving the quadratic equation obtained as $\det(A - sI) = s^2 - \mathrm{tr}(A)s + \det(A) = 0$, which in this case is

$$s^2 - 4s + 3 = (s-1)(s-3) = 0.$$

The roots are then $s_1 = 1$ and $s_2 = 3$. We now find the eigenvectors.

$$\boxed{s_1 = 1}: \quad Ab_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} = s_1 b_1 = \begin{pmatrix} h \\ k \end{pmatrix},$$

which gives $2h + k = h$ and $h + 2k = k$, either of which give $h = -k$. Thus, $b_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$$\boxed{s_2 = 3}: \quad Ab_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} = s_2 b_2 = 3\begin{pmatrix} h \\ k \end{pmatrix},$$

which gives $2h + k = 3h$ and $h + 2k = 3k$, either of which give $h = k$. Thus, $b_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. This gives then $v_1(t) = e^{r_1 t}b_1 = e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and $v_2(t) = e^{r_2 t}b_2 = e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Defining now the fundamental solution matrix $\mathcal{X}(t)$ as

$$\mathcal{X}(t) = \begin{pmatrix} v_1(t) & v_2(t) \end{pmatrix} = \begin{pmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{pmatrix},$$

we have that $v_h = \mathcal{X}(t)c$, with $c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

(b) (10 points) Find a particular solution $v_p = \mathcal{X}(t)w(t)$ using variation of parameters.

SOLUTION: There is no reason to rederive the formula for $w(t)$ here; we know that doing variation of parameters (plugging v_p back into the ODE) will always produce the following ODE for the unknown $w(t)$:

$$\mathcal{X}(t)w' = f(t), \quad \text{which is}: \quad \begin{pmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{pmatrix}\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} e^t \\ -e^t \end{pmatrix}.$$

We must solve for $w_1(t)$ and $w_2(t)$. Adding the two equations together gives $w_2' = 0$, so $w_2(t) = 1$ (any constant works; we can just make the choice of $C = 1$). We now plug this into the first equation to get $w_1' = 1$, or $w_1(t) = t$. Thus, $v_p(t) = \mathcal{X}(t)w(t)$, with $w(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}$.

(c) (5 points) Find the solution $v(t)$ corresponding to the given initial data $v(0)$.

SOLUTION: Just write down the final general solution in the form $v(t) = v_h(t) + v_p(t) = \mathcal{X}(t)(c + w(t))$ and then use the initial data $v(0)$ to determine c. I.e., $v(0) = \mathcal{X}(0)(c + w(0))$, which is

$$v(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} e^0 & e^{3(0)} \\ -e^0 & e^{3(0)} \end{pmatrix}\left\{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\left\{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\},$$

which gives $c_1 + c_2 = -1$, $c_2 - c_1 = 0$, or $c_1 = c_2 = -\frac{1}{2}$. This gives then finally $v(t) = \mathcal{X}(t)(c + w(t))$, where $c = \begin{pmatrix} -1/2 \\ -1/2 \end{pmatrix}$. To verify we have it right, compute $v'(t)$, and verify that it is the same as $Av(t) + f(t)$.