\[ x' = 1 + x + x^2 \cos t = f(x, t). \quad (1) \]
\[ x(0) = 0. \]

For \((t, x) \in \left[-\frac{1}{3}, \frac{1}{3}\right] \times [-\beta, \beta]\)
we notice that
\[ |f(x, t)| \leq 1 + |x| + |x^2| |\cos t| \leq 1 + \beta + \beta^2. \]

i.e. \[ |f(x, t)| \leq M = 1 + \beta + \beta^2. \]

Let \( R = \{ (t, x): |t| \leq \frac{1}{3}, |x| \leq \beta \} \).

Then by Theorem 8.1.1 (1) has a solution
for \(|t| \leq \min\left(\frac{1}{3}, \frac{\beta}{1 + \beta + \beta^2}\right)\). So the question
of existence boils down to whether we can choose \( \beta > 0 \) s.t.
\[ \frac{\beta}{1 + \beta + \beta^2} \geq \frac{1}{3}. \]

One can easily see that \( \beta = 1 \) satisfies the derived inequality.
\[ x' = x^2 \]
\[ x(0) = 1. \]

Our rectangle is defined as:
\[ R = \{ (t, x) : |t| \leq \alpha, \quad \frac{-\beta}{x - 1} \leq \beta \}. \]

We need to find how large we can make the quantity
\[ \min \{ \alpha, \frac{\beta}{M} \} \]
where \( M = \max_{(t,x) \in R} |f(x,t)|. \) So we do some calculations. For \( \beta \leq \alpha, \quad |x - 1| \leq \beta \) we have:
\[ x - 1 \leq \beta \quad \text{and} \quad 1 - x \leq \beta. \]

or just \( x \leq \beta + 1 \) (\( \beta \) is always positive).

Then:
\[ |f(x,t)| = x^2 \leq (\beta + 1)^2 = M. \]

Also, how large can \( M \) be?

\[ \frac{\beta}{M} = \frac{\beta}{(\beta+1)^2} \]

and to maximize the interval of existence.

I want to choose \( \beta > 0 \) s.t. \( \frac{\beta}{(\beta+1)^2} \) is maximized. Turns out this is done by solving a simple calculus problem for maximizing
\[ g(\beta) = \frac{\beta}{(\beta+1)^2} \]
we get that \( \max_{\beta > 0} g(\beta) = \frac{1}{4} \) for \( \beta = 1. \)
Thus, if we take $\beta = 1$, \[ \frac{\beta}{(\beta + 1)^2} = \frac{1}{4} = \frac{\beta}{M} \]
and take $x = \frac{1}{4}$ we have existence on the interval $|t| \leq \frac{1}{4}$.

Theorem 8.1.2 would predict the same interval of existence, but also more uniqueness.

Theorem 8.1.3 requires $f$ to be globally Lipschitz (i.e. with no condition on the size of $x$) which we cannot achieve for
\[ f(x, t) = x^2. \]

Since $x^2$ is not a globally Lipschitz function.

The actual solution to our ODE is
\[ x(t) = \frac{1}{1 - t} \]
which is finite for any closed interval contained in $(-\infty, 1)$.

**8.2.2**

\[ x' = x^{1/2} \]
\[ x(0) = 0. \]

Such \[ \frac{d}{dt} \left( \frac{t^2}{4} \right) = \frac{t}{2} = \sqrt{\frac{t^2}{4}} \]
and \[ \left. \frac{t^2}{4} \right|_{t=0} = 0 \] we see \[ x(t) = \frac{t^2}{4}. \]
is an analytical solution to our ODE.

The first order Taylor's method, also commonly known as Euler's method, for our problem is

\[ x(t+h) = x(t) + h \sqrt{x(t)} \]

Starting at \( t=0 \), \( x(0) = 0 \) and we get

\[ x(h) = 0 + h \sqrt{0} = 0. \]

Then, \( x(2h) = 0 + h \sqrt{0} = 0 \), which any we see that we will always get zero from this method.

The key thing to note here is that our ODE does not have a unique solution, and Euler's method is picking out the other solution \( x(t) = 0 \).

8.2.8 \[ x' = f(x,t). \]

\[ x'' = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} = f_t + f_x \cdot \dot{x}. \]

\[ x''' = f_{tt} + f_{tx} \cdot \dot{x} + f_{xt} \cdot \ddot{x} + f_{xx} \cdot \dddot{x} + f_x \cdot (f_t + f_x \cdot \dot{x}). \]

\[ = f_{tt} + 2 f_{xt} \cdot \dddot{x} + f_{x} f_{xx} \ddot{x} + f_x \dddot{x} + f_x f_t + f_x^2 f. \]
\[ x''' = f_{ttt} + f_{ttx} \cdot \dot{x} + 2(f_{xtc} + f_{xtx} \cdot \dot{f}) \cdot \dot{x} \]
\[ + 2f_{xt}(\dot{f}_t + f_x \cdot \dot{f}) + (f_{xxt} + f_{xxx} \cdot \dot{f}) \cdot \dot{x}^2 \]
\[ + f_{xx} \cdot 2\dot{f} \cdot (\dot{f}_t + f_x \cdot \dot{f}) + \frac{f_{xxx} \cdot \dot{f}}{f_x} \cdot \dot{f} \]
\[ + (f_{xt} + f_{xx} \cdot \dot{f}) \ddot{f}_t + f_x (f_{tt} + f_{tx} \cdot \dot{f}) \]
\[ + 2f_x \cdot (f_{xt} + f_{xx} \cdot \dot{f}) \ddot{f} + f_x^2 \cdot (\ddot{f}_t + f_x \cdot \ddot{f}) \cdot \dot{f} \]

(2 think this is correct ...)

8.3.1

The second order Runge-Kutta method is:
\[ x(t+h) = x(t) + \omega_1 h f(x,t) + \omega_2 h \frac{f(x,t+\alpha h)}{1} + O(h^3) \]

\[ \text{With } \omega_1 + \omega_2 = 1 \]
\[ \omega_2 \alpha = \frac{1}{2} \]
\[ \omega_2 \beta = \frac{1}{2} \]

Since \( \alpha = \frac{2}{3} \) we get \( \beta = \frac{2}{3} \) and \( \omega_2 = \frac{3}{4} \)

\[ \Rightarrow \omega_1 = \frac{1}{4} \]

Thus our formula is:
\[ x(t+h) = x(t) + \frac{1}{4} h f(x,t) + \frac{3}{4} h f(x + \frac{2}{3} h f(x,t), t+\frac{2}{3} h) \ldots \]