7.2.18 and 19.

I'm not sure what #19 wants different, so I'll just derive the composite rule for \( \int_a^b f(x) \, dx \) for unequal spacing.

Let \( a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b \).

\[
\int_a^b f(x) \, dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x) \, dx.
\]

Now to approximate \( \int_{x_i}^{x_{i+1}} f(x) \, dx \), we transform to an integral over \([-1, 1]\) and use the given midpoint rule.

\[
\int_{x_i}^{x_{i+1}} f(x) \, dx = \int_{-1}^{1} f(x_i + \frac{x_{i+1} - x_i}{2}(y+1)) \frac{(x_{i+1} - x_i)}{2} \, dy,
\]

\[
y = 2\frac{x - x_i}{x_{i+1} - x_i} - 1
\]

\[
dy = \frac{2}{x_{i+1} - x_i} \, dx.
\]

\[
x = \frac{(x_{i+1} - x_i)}{2}(y+1) + x_i
\]

\[
\approx (x_{i+1} - x_i) f\left( x_i + \frac{x_{i+1} - x_i}{2} \right)
\]

\[
= (x_{i+1} - x_i) f\left( \frac{x_i + x_{i+1}}{2} \right)
\]
Thus one composite rule for \( \int_a^b f(x) \, dx \)

is \( \int_a^b f(x) \, dx \approx \sum_{i=0}^{n-1} (x_{i+1} - x_i) f \left( \frac{x_i + x_{i+1}}{2} \right) \)

2) \( x_i = a + \frac{b-a}{n} i \)  (i.e. equal spacing)

\[
\begin{align*}
    x_{i+1} - x_i &= \frac{b-a}{n} \\
    \frac{x_{i+1} + x_i}{2} &= a + \frac{b-a(i+1)}{n} + a + \frac{b-a}{n} i \\
    &= a + \frac{b-a}{n} (i+\frac{1}{2})
\end{align*}
\]

So the rule for equal spacing is

\[
\int_a^b f(x) \, dx \approx \frac{b-a}{n} \sum_{i=0}^{n-1} f \left( a + \frac{b-a}{n} (i+\frac{1}{2}) \right).
\]

7.3.7

a) Let \( (p, q) = \int \int p(x)q(x) \, dx \), \( n=1 \), we want to find \( g \in P_2 \) which is orthogonal (w.r.t our above inner-product) to all polynomials in \( P_1 \). To do this, we perform Gram-Schmidt in \( \mathbb{R}[x] \) w.r.t our inner-product.
Let \( q_0(x) = 1 \)

\[
q_1(x) = x - \frac{\langle x, q_0 \rangle}{\langle q_0, q_0 \rangle} q_0.
\]

\[
q_2(x) = x^2 - \frac{\langle x^2, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1 - \frac{\langle x, q_0 \rangle}{\langle q_0, q_0 \rangle} q_0.
\]

\[
\langle x, q_0 \rangle = \int_0^1 x \cdot 1 \cdot 1 = \int_0^1 x^2 dx = \frac{1}{3}.
\]

\[
\langle q_0, q_0 \rangle = \int_0^1 x \cdot 1 \cdot 1 = \int_0^1 x dx = \frac{1}{2}.
\]

\[
\Rightarrow q_1(x) = x - \frac{\frac{1}{3}}{\frac{1}{2}} = x - \frac{2}{3}.
\]

\[
\langle x^2, q_1 \rangle = \int_0^1 x \cdot x^2 (x - \frac{2}{3}) dx = \frac{1}{36}.
\]

\[
\langle q_1, q_1 \rangle = \int_0^1 x \cdot x^2 \cdot 1 = \frac{1}{4}.
\]

\[
K.
\]

\[
\Rightarrow q_2(x) = x^2 - \frac{\frac{1}{36}}{\frac{1}{4}} (x - \frac{2}{3}) = \frac{1}{2}
\]

\[
= x^2 - \frac{6}{5} x + \frac{3}{10}.
\]

Now let \( q = q_2 \). Since then \( q \) is orthogonal to \( q_0, q_1, \) and \( q_2 \), which span the polynomials \( P_1 \), we have what we want.
Now we take our quadrature points to be the zeros of $g$ which are

$$x_0 = \frac{3}{5} + \frac{\sqrt{6}}{10}, \quad x_1 = \frac{3}{5} - \frac{\sqrt{6}}{10}$$

Now we need to find $A_0$ and $A_1$ so that

$$\int x f(x) \, dx \approx A_0 f(x_0) + A_1 f(x_1)$$

is exact for all polynomials in $P_3$.

To do so, we use the method of undetermined coefficients.

$$\frac{1}{2} = \int x \cdot 1 \, dx = A_0 \cdot 1 + A_1 \cdot 1$$

$$\frac{1}{3} = \int x \cdot x \, dx = A_0 x_0 + A_1 x_2.$$

Solving (via MATLAB)

$$A_0 = \frac{1}{4} + \frac{\sqrt{6}}{36}, \quad A_1 = \frac{1}{4} - \frac{\sqrt{6}}{36}.$$

So our quadrature rule is

$$\int x f(x) \, dx \approx \left(\frac{1}{4} + \frac{\sqrt{6}}{36}\right) f\left(\frac{3}{5} + \frac{\sqrt{6}}{10}\right) + \left(\frac{1}{4} - \frac{\sqrt{6}}{36}\right) f\left(\frac{3}{5} - \frac{\sqrt{6}}{10}\right)$$

which you can check to be exact for all polynomials of the form

$$p(x) = ax^3 + bx^2 + cx + d.$$
b. Since \( n=2 \) we need to find a third order polynomial which is orthogonal to \( \{1, x, x^2\} \) in the previous \( \{g_0, g_1, g_2\} \) span \( P_2 \) we just need to perform one more step of the Gram-Schmidt.

\[
g_3 (x) = x^3 - \frac{(x^3, g_2)}{(g_2, g_2)} g_2 - \frac{(x^3, g_1)}{(g_1, g_1)} g_1 - \frac{(x^3, g_0)}{(g_0, g_0)} g_0.
\]

\[
(x^3, g_2) = \int_0^1 x \cdot x^3 \cdot (x^2 - \frac{6}{5} x + \frac{3}{10}) \, dx = \frac{1}{350}.
\]

\[
(g_2, g_2) = \int_0^1 x (x^2 - \frac{6}{5} x + \frac{3}{10})^2 \, dx = \frac{1}{600}.
\]

\[
(x^3, g_1) = \int_0^1 x \cdot x^3 (x - \frac{2}{3}) \, dx = \frac{1}{30}.
\]

\[
(g_1, g_1) = \int_0^1 x^3 \cdot 1 \, dx = \frac{1}{5}.
\]

\[
g_3 (x) = x^3 - \frac{600}{350} \left( x^2 - \frac{6}{5} x + \frac{3}{10} \right) - \frac{36}{30} (x - \frac{2}{3}) - \frac{2}{5}.
\]

\[
= x^3 - \frac{12}{7} x^2 + \frac{6}{7} x - \frac{4}{35}.
\]

Let \( g = g_3 \), and now we find the zeros of \( g \) (we MATLAB)

\[
x_0 \approx 0.911412
\]

\[
x_1 \approx 0.2123405
\]

\[
x_2 \approx 0.590533
\]
Now we have the rule:
\[
\int_0^1 xf(x) \, dx \approx A_0 f(x_0) + A_1 f(x_1) + A_2 f(x_2)
\]

Plugging in \( f(x) = 1, x, \) and \( x^2 \) we find that
\[
A_0 \approx 0.2099319 \\
A_1 \approx 0.0698269 \\
A_2 \approx 0.22924111
\]

7.2.20

Reproduce the steps for 7.2.7 but you don't have to find the zeros of \( q \). Also, the new inner product should be given by
\[
(p, q) = \int_{-1}^{1} (1 + x^2) p(x) q(x) \, dx.
\]

Find \( \xi_0, \xi_1, \xi_2, \xi_3 \) using Gram-Schmidt on \( \xi_1, x, x^2, x^3 \). Take \( \xi = \xi_3 \).
\[ S(f, h) - I = c_4 h^4 + c_6 h^6 + \ldots \]
\[ S(f, \frac{h}{3}) - I = c_4 \frac{h^4}{3^4} + c_6 \frac{h^6}{3^6} + \ldots \]
\[ 3^4 S(f, \frac{h}{3}) - 3^4 I = c_4 h^4 + c_6 h^6 + \ldots \]
\[ S(f, h) - 3^4 S(f, \frac{h}{3}) - (1 - 3^4) I = \frac{8 c_6}{9} h^6 + \ldots \]

\[ \Rightarrow \frac{1}{(1 - 3^4)} \left[ S(f, h) - 3^4 S(f, \frac{h}{3}) \right] - I \]

\[ = \frac{8 c_6}{9(1 - 3^4)} h^6 + \ldots \]

Thus, if our new quadrature rule is given by
\[ \int_a^b f(x) \, dx = I \approx \frac{1}{(1 - 3^4)} \left[ S(f, h) - 3^4 S(f, \frac{h}{3}) \right] \]

it is \( O(h^6) \) accurate.