1. We say that an ODE is well-posed if we have the following three conditions satisfied:

(i) Existence
(ii) Uniqueness
(iii) Stability with respect to initial data

We have a theorem which says that for

the ODE

\[ y' = f(t, y), \quad t \in [a, b] \]
\[ y(a) = \xi \]

the ODE will be well-posed if \( f \) is Lipschitz in the strip

\[ D = \{ (t, y) : a \leq t \leq b, -\infty < y < \infty \} \]

This Lipschitz condition can be verified by showing

\[ \left| \frac{\partial f}{\partial y}(t, y) \right| \]

is uniformly bounded.

For our problem

\[ f(t, y) = 1 - 5y + e^t \]

\[ \frac{\partial f}{\partial y} = -5 \quad \Rightarrow \quad \left| \frac{\partial f}{\partial y} \right| \leq 5 \]

Thus we get well-posedness.
2. \( y''' = y'' + 5y' - 2y \)
   \( y(0) = y'(0) = 0 \quad y''(0) = 1 \)

Let
   \( x_1 = y \)
   \( x_2 = y' = x_1' \)
   \( x_3 = y'' = x_2' \)

Then we have
   \( x_1' = x_2 \)
   \( x_2' = x_3 \)
   \( x_3' = x_3 + 5x_2 - 2x_1 \)

with initial conditions \( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \).
3. (a) Forward Euler with \( h = 0.5 \) and two time steps:

\[
y(\frac{1}{2}) = y(0) + \frac{1}{2} f(0, y(0)).
\]

\[
y(1) = y(\frac{1}{2}) + \frac{1}{2} f(\frac{1}{2}, y(\frac{1}{2})).
\]

where \( y(0) = 1 \).

\[
y(\frac{1}{2}) = 1 + \frac{1}{2} f(0, 1) = 1 + \frac{1}{2} \{ 1 - 5 \cdot 1 + e^0 \}
\]

\[
= \frac{-1}{2}
\]

\[
y(1) = \frac{-1}{2} + \frac{1}{2} f(\frac{1}{2}, -\frac{1}{2}) = \frac{-1}{2} + \frac{1}{2} \{ 1 - 5 \left( -\frac{1}{2} \right) + e^{-\frac{1}{2}} \}
\]

\[
= \frac{1}{2} \left[ -1 + 1 + \frac{5}{2} + e^{\frac{-1}{2}} \right].
\]

\[
= \frac{5}{4} + \frac{1}{2} e^{\frac{-1}{2}} \text{ approx solution at } t = 1
\]

(b) The truncation error is the difference between an approximation of the derivative and the actual derivative. That is,

\[
y_{k+1} - y_k - y'(t_k) = \text{ truncation error}
\]

(c) For forward Euler we have, by Taylor:

\[
y(t_{k+1}) = y(t_k) + y'(t_k) h + \frac{y''(t_k) h^2}{2} + o(h^3)
\]

For Euler we have:

\[
y_{k+1} - y_k - y'(t_k) = \frac{1}{2} y''(t_k) h^2 + o(h^2)
\]
4. \[\omega_0 = \alpha, \quad \omega_{i+1} = \omega_i + h \left[ \Theta f(t_i, \omega_i) + (1-\Theta) f(t_{i+1}, \omega_{i+1}) \right]\]

(a) \(\Theta = 1\) gives us:
\[\omega_{i+1} = \omega_i + h f(t_i, \omega_i)\]

with the polynomials \(p(\tau) = \tau - 1\), \(\varepsilon(\tau) = 1\).

The root of \(p\) is \(\tau = 1\) and thus the method is stable.

Since \(p(1) = 0\) and \(p'(1) = 1 = \varepsilon(1)\)
the method is also consistent.

To determine the region of stability we analyze the one-step method for the ODE
\[y' = 2y, \quad y(0) = \frac{1}{2}\]
Then the one-step method is
\[y_{k+1} = y_k + h 2y_k\]
We want to find a range for \(h\) s.t. \(y_k \to 0\) as \(k \to \infty\). To do so, we have the recursion formula:
\[y_k = (1 - h|\lambda|)^k y_0\]

and in order for \(y_k \to 0\) we need
\[|1 - h| \lambda| < 1\]

\[\Rightarrow h < \frac{2}{|\lambda|}\]

This is our region of stability.
(b) For this for \( \theta = 0 \) we have

\[ w_{i+1} = w_i + hf(t_i, w_i) \]

giving us the polynomials

\[ p(z) = z - 1 \quad \text{and} \quad q(z) = z. \]

One can then verify, the method is stable and consistent.

To get the region of stability we again examine the simple ODE

\[ y' = 2y \quad t < 0 \]

and our one-step method gives us

\[ y_{k+1} = y_k + h2y_k \]

or

\[ y_{k+1} = \frac{1}{1 + h\lambda_1} y_k. \]

But since \( h\lambda_1 \) are positive, clearly

\[ y_k \to 0. \]

Thus for \( \theta = 0 \) \((0, \infty)\) is our region of stability.