MT 1 Solutions (somewhat...)

* $P_2(x) = f(x_0) \xi_0(x) + f(x_1) \xi_1(x) + f(x_2) \xi_2(x)$

where

\begin{align*}
\xi_0(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{1}{2} (x-1)(x-2) \\
\xi_1(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = -x(x-2) \\
\xi_2(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{1}{2} x(x-1)
\end{align*}

* Plug \( \frac{1}{2} \) into $P_2$ to approximate $f$ at $x = 1/2$.

* The general form for the error is given by

\[
|f(x) - P_2(x)| = \frac{1}{(2+1)!} f^{(3)}(\xi_x) \cdot x(x-1)(x-2)
\]

where $\xi_x \in (a,b)$.

* If $f(x) = 2x^3 - x^2 + 1$ then

\[
f^{(3)}(x) = 12
\]

and

\[
|f(x) - P_2(x)| = \left| \frac{12}{6} \cdot x(x-1)(x-2) \right| \quad x \in [0, 2].
\]

\[
\leq 2 \cdot 2 \cdot 1 \cdot 2 = 8.
\]

(\( \xi \) This is a bad estimate, but it works)
2. **Taylor Expansion** says:

\[
f(x + h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{6}f'''(x)h^3 + o(h^4)
\]

\[
f(x + 2h) = f(x) + 2f'(x)h + 2f''(x)h^2 + \frac{4}{3}f'''(x)h^3 + o(h^4)
\]

Thus:

\[
f(x) - 2f(x + h) + f(x + 2h)
\]

\[
= f(x) - 2f(x) - 2f'(x)h - f''(x)h^2 - \frac{1}{3}f'''(x)h^3 + f(x) + 2f'(x)h + 2f''(x)h^2 + \frac{4}{3}f'''(x)h^3 + o(h^4).
\]

i.e.

\[
\frac{f(x) - 2f(x + h) + f(x + 2h)}{h^2} = f''''(x) + o(h^2).
\]

Thus an expression for the error is given by:

\[
\sqrt[5]{f'''(\xi_x)h}
\]

for some \( \xi_x \in [0, h] \).

\[
\frac{f''(0)}{1} \approx \frac{f(0) - 2f(1) + f(2)}{1^2} = \frac{1 - 4 + 13}{1} = 10.
\]
3. Trapezoidal Rule
\[
\int_{0}^{2} x^4 \, dx \approx \left[ \frac{1}{2} (0)^4 + \frac{1}{2} (2)^4 \right] \cdot (2-0) = 16.
\]

Simpson's Rule
\[
\int_{0}^{2} x^4 \, dx \approx \frac{2-0}{6} \left[ \frac{8}{2} (0)^4 + 4 (1)^4 + (2)^4 \right] = \frac{1}{3} [4 + 16] = \frac{20}{3}.
\]

The error term for the Trapezoidal rule is given by
\[
-\frac{1}{12} (b-a)^3 \frac{f''(\xi)}{2} \in (a, b).
\]

For Simpson's rule,
\[
-\frac{1}{90} \left[ \frac{(b-a)}{2} \right]^5 f^{(4)}(\xi) \in (a, b).
\]

For our particular choice of \( f(x) = x^4 \) we have for the Trap rule;
\[
-\frac{1}{12} \cdot 2^3 \cdot 8^2 = -88^2 \quad \text{Error bound for trap}
\]

i.e. \( |\text{error}| \leq 8 \cdot 2^2 = 32 \).

For Simpson
\[
-\frac{1}{90} \left[ \frac{2}{2} \right]^5 f^{(4)}(\xi) = -\frac{1}{90} \cdot 16 \cdot 2^4
\]
i.e. a bound for the error is
\[ |\text{error}| \leq \frac{24}{90}. \]

Now, the exact integral is
\[ \int_0^2 x^4 \, dx = \frac{1}{5} x^5 \bigg|_0^2 = \frac{1}{5} \cdot 2^5 = \frac{32}{5}. \]

The approximate Trap rule was.
\[ |16 - \frac{32}{5}| < 3.2 < 1. \]

Using Simpson's rule.
\[ \left| \frac{20}{3} - \frac{32}{5} \right| = \frac{24}{90} = \text{Exactly the predicted error}. \]

We want to now consider the composite rules which are given by
\[ \frac{-1}{12} (b-a) h^2 f''(c) \quad \text{and} \quad \frac{-1}{180} (b-a) h^4 f^{(4)}(c). \]

Want the composite error less than \( 10^{-5} \)

That is
\[ \left| \frac{-1}{12} \cdot \frac{1}{2} h^2 \cdot \frac{32}{12} \right| \leq 10^{-5}. \]

And
\[ \left| \frac{-1}{180} h^4 \cdot 24 \right| \leq 10^{-5} \]
\[ \Rightarrow \left| \frac{16}{3} h^2 \right| \leq 10^{-5} \]
\[ \Rightarrow h = \sqrt{\frac{180}{48}} \cdot 10^{-5} \]
\[ \Rightarrow h = \sqrt{\frac{3}{16}} \cdot 10^{-5} \]
4. \( f'(x) = \left[ \frac{f(x+h) - f(x)}{h} \right] - \frac{f''(x)}{2} h - \frac{f'''(x)}{6} h^2 + O(h^3) \)

is written as

\[ M = N(h) + k_1 h + k_2 h^2 + O(h^3). \]

We want to find a, b, c, s.t. if we put

We also have

\[ M = N(2h) + 2k_1 h + 4k_2 h^2 + O(h^3) \]

\[ M = N(3h) + 3k_1 h + 9k_2 h^2 + O(h^3). \]

If we then multiply each equation by a, b, c and add we have:

\[ (a+b+c)M = \text{const.} \]

\[ aN(h) + bN(2h) + cN(3h) \]

\[ + (a+2b+3c)k_1 h \]

\[ + (a+4b+9c)k_2 h^2 \]

\[ + O(h^3). \]

We get what we want if the following hold:

\[ a + b + c = 1 \]

\[ a + 2b + 3c = 0. \]

\[ a + 4b + 9c = 0. \]
and then,

\[ aN(h) + bN(2h) + cN(3h) \] would be our new approx with error \( O(h^3) \).

Solving the above system for \( a, b, c \) we get

\[ a = 3 \quad b = -3 \quad c = 1. \]