FUNCTIONAL ANALYSIS

With Applications in Numerical Analysis

(Lecture notes for AMa 204)

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Contents of the Lecture Notes

4. **Linear and Nonlinear Operators**

   (a) Mappings of spaces and the four fundamental subspaces
   (b) Linear operators as mappings of spaces, inverse mappings
   (c) Continuity of the norm and inner-product as operators on normed and inner-product spaces
   (d) Continuity and linear operators; boundedness, continuity at a point
   (e) The bound of a linear operator and the operator norm
   (f) The (Hilbert) adjoint operator, self-adjoint operators
   (g) Identity, null, invertible, isometric, and positive operators
   (h) Compact operators
   (i) Projection operators
   (j) Bounded and unbounded operators
   (k) An example of a bounded linear operator: an integral operator
   (l) An example of an unbounded linear operator: a differential operator
   (m) Linear and bilinear forms: continuity (boundedness), coercivity, symmetry, positivity
   (n) The operator norm of linear and bilinear forms
   (o) Quadratic forms and the Polarization Identity
   (p) The Equality of Forms Theorem
   (q) The Bounded Bilinear Form Theorem
   (r) The Bounded Operator Theorem
   (s) The Lax-Milgram Theorem
   (t) The Lions-Stampachia Theorem, and the relationship to Lax-Milgram and minimization
   (u) Nonlinear operators: first variation, G(Gateaux)-variation, G-differential, G-derivative
   (v) The F(Frechet)-differential, F-derivative, Gradients and potential mappings
   (w) Calculating G-derivatives and F-derivatives of nonlinear operators
   (x) Euler Conditions
   (y) Monotone operators and the Nonlinear Lax-Milgram Theorem

5. **Additional Topics in Functional Analysis**

   (a) Dual spaces again, duality pairing, isomorphisms and isometries
   (b) Gelfand Triples and the pivot space
   (c) Extensions of operators and forms
   (d) Continuous and compact operators ("completely continuous" operators)
   (e) Continuous and compact imbeddings of abstract spaces, imbedding operators
   (f) Compactly imbedded subspaces of a Hilbert spaces and density
   (g) Intermediate spaces of Hilbert (or Banach) spaces
   (h) Interpolation inequalities and interpolation spaces
   (i) Hilbert scales of spaces
   (j) The spectral theory of self-adjoint linear operators
   (k) The Hilbert-Schmidt Theorem
   (l) The Spectral Theorem
   (m) The Hahn-Banach Theorem
   (n) The Closed-Graph Theorem (and the Open Mapping Theorem)
   (o) The Principle of Uniform Boundedness

6. **Measure and Integration**

   (a) Set functions and the Banach-Tarski Paradox
   (b) Borel fields, σ-algebras, and measurable sets
   (c) Measures and the Lebesgue measure in $\mathbb{R}$ and $\mathbb{R}^n$
   (d) Open sets $\Omega \in \mathbb{R}^n$ as domains of functions
   (e) Measurable functions
   (f) Sets of measure zero and the "almost everywhere" (AE) notion
   (g) Step functions, characteristic functions, simple functions
   (h) The Lebesgue and Riemann integrals; Lebesgue and Riemann integrable functions
   (i) Fatou's lemma and the Monotone Convergence Theorem
   (j) Lebesgue's Dominated Convergence Theorem
   (k) The $L^p(\Omega)$ spaces as equivalence classes of functions, and the $L^p(\Omega)$ norms
   (l) Young's inequality, Holder's inequality, and Minkowski's inequality
   (m) The analogous discrete ℓ^p spaces and analogous inequalities
   (n) The $L^p(\Omega)$ spaces are Banach spaces (the Riesz-Fischer Theorem)
   (o) The special case of the Hilbert space $L^2(\Omega)$ and its inner-product

7. **Distributions, Weak Derivatives, and Sobolev Spaces**

   (a) Some additional function spaces ($C^k(\Omega)$, $\bar{C}^k(\Omega)$, $C_0^k(\Omega)$, $\ldots$) and their (Banach) norms
   (b) The (Schwartz) Theory of Distributions
   (c) "Test" functions and the spaces $L^1_{loc}(\Omega)$, $\mathcal{D}(\Omega)$, $\mathcal{D}'(\Omega)$
   (d) Distributions as bounded linear functionals in $\mathcal{D}'(\Omega)$ over the space $\mathcal{D}(\Omega)$
   (e) Conditions for sets Ω: the cone condition, Lipshitz continuous boundaries, the "space" $\mathcal{C}^{0,1}$
   (f) The "weak derivative" and the space of k-times weakly differentiable functions $W^k(\Omega)$
   (g) The Sobolev spaces $W^{k,p}(\Omega)$ based on $W^k(\Omega)$ and $L^p(\Omega)$
   (h) The norms and semi-norms in the spaces $W^{k,p}(\Omega)$ and the myriad of common notations
   (i) The spaces $W^{k,p}(\Omega)$ are Banach spaces (no proof; pointers to Adams)
   (j) The special Hilbert space case of $p = 2$: $H^k(\Omega) = W^{k,2}(\Omega)$, and the inner-product

8. **The Sobolev Imbedding, Compactness, Density, Extension, and Trace Theorems**

   (a) Review of the the formal "completion" of an arbitrary metric space
   (b) The Sobolev spaces $H^{k,p}(\Omega)$ based on the completion of $C^k(\Omega)$ in the $W^{k,p}(\Omega)$ norm
   (c) $H^{k,p}(\Omega) = W^{k,p}(\Omega)$ (the Meyers-Serrin Theorem; no proof, pointer to Adams)
   (d) Continuous and compact operators; continuous and compact imbeddings of abstract spaces
   (e) The Sobolev Imbedding Theorems and implications for the finite element interpolant
   (f) Compact Imbeddings, Density Theorems, and Extension Theorems
   (g) The Sobolev Integral Identity and Poincaré-like inequalities
   (h) Generalizing results (e.g., the Divergence Theorem) to Sobolev spaces using density arguments
   (i) The Trace Theorem and fractional order Sobolev spaces
   (j) Variational formulation of elliptic partial differential equations
   (k) The problem of delta functions: bounded linear functionals and Sobolev imbedding theorems
   (l) **Extended Example:** An analysis of the linearized and nonlinear Poisson-Boltzmann equations

Reference Material

Below is a list of what I feel are some of the most important reference books for the “intersection” area of functional analysis with numerical analysis and the modern theory elliptic partial differential equations.

- General numerical analysis: [10, 11, 22, 40, 48]
- General numerical treatment of elliptic equations: [18, 20, 43]
- Finite element theory: [3, 4, 6, 9, 18, 39, 42]
- Finite element implementation: [3, 5, 23]
- Iterative methods for linear problems: [19, 49, 50, 52]
- Iterative methods for nonlinear problems: [13, 25, 35, 41]
- Multigrid and domain decomposition methods: [8, 17, 46]
- Linear elliptic equations: [2, 15, 18, 34, 37, 44, 47]
- Nonlinear elliptic equations: [14, 15, 38]
- Real analysis: [24, 28, 36, 45]
- Functional analysis: [7, 12, 25, 27, 33, 51]
- Operator and matrix theory [16, 19, 21, 26, 29, 49]
- Sobolev and Besov spaces: [1, 30, 31, 32]

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