Some Research Problems in GR

Michael Holst

Einstein Equations
Einstein Equations
Conformal Method
Results: 1973–2007
1973–1995: CMC
1996–2007: Near-CMC

Conformal Theory
Far-From-CMC (2008–)
Rough Metrics (2009–)
Compact Case (2013–)
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Algorithms
AFEM Convergence
FEEC with AFEM
FEEC for Evolution

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Some Research Problems in Mathematical and Numerical General Relativity

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NSF DMS/FRG 1262982: Analysis of the Einstein Constraint Equations
NSF DMS/CM 1620366: Numerical Methods for Geometric PDE

Some of my lecture is from:

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THE EMERGENCE OF GRAVITATIONAL WAVE SCIENCE: 100 YEARS OF DEVELOPMENT OF MATHEMATICAL THEORY, DETECTORS, NUMERICAL ALGORITHMS, AND DATA ANALYSIS TOOLS

MICHAEL HOLST, OLIVIER SARBACH, MANUEL TIGLIO, AND MICHELE VALLISNERI

In memory of Sergio Dain

Abstract. On September 14, 2015, the newly upgraded Laser Interferometer Gravitational-wave Observatory (LIGO) recorded a loud gravitational-wave (GW) signal, emitted a billion light-years away by a coalescing binary of two stellar-mass black holes. The detection was announced in February 2016, in time for the hundredth anniversary of Einstein's prediction of GWs within the theory of general relativity (GR). The signal represents the first direct detection of GWs, the first observation of a black-hole binary, and the first test of GR in its strong-field, high-velocity, nonlinear regime. In the remainder of its first observing run, LIGO observed two more signals from black-hole binaries, one moderately loud, another at the boundary of statistical significance. The detections mark the end of a decades-long quest and the beginning of GW astronomy: finally, we are able to probe the unseen, electromagnetically dark Universe by *listening* to it. In this article, we present a short historical overview of *GW science*: this young discipline combines GR, arguably the crowning achievement of classical physics, with record-setting, ultra-low-noise laser interferometry, and with some of the most powerful developments in the theory of differential geometry, partial differential equations, high-performance

Outline (Starting Part 1)

1 The Einstein Equations
- Einstein Evolution and Constraint Equations
- The Conformal Method(s) of 1944, 1973, 1974
- Analysis of the Conformal Method: 1973–1995, 1996–2007
- The 1973–1995 CMC Results
- The 1996–2007 Near-CMC Results

2 New Conformal Method Results from our Group (2008-2019)
- First Results for Far-From-Constant Mean Curvature (2008–)
- Results for Rough Metrics (2009–)
- Compact with Boundary Case (2013–)
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- Multiplicity through Analytic Bifurcation Theory (2017–)

3 Design and Analysis of Approximations and Algorithms
- AFEM Convergence with Goal Functions
- AFEM Convergence in the FEEC Setting
- FEEC A Priori Error Estimates for Evolution Problems

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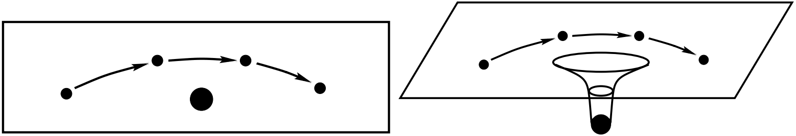
General Relativity and Einstein's Equations

Einstein's 1915 ***general theory of relativity*** states what we experience as gravity is simply the ***curvature of our spacetime*** when it is viewed as a geometrical object $\mathcal{M}$, known as a *pseudo-Riemannian manifold*.

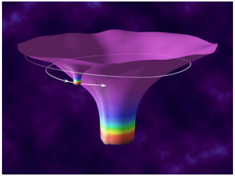
Newtonian vs. General Relativistic Theories of Gravity:

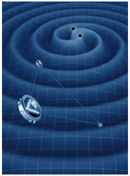


Curvature in our ***spacetime*** $\mathcal{M}$ is governed by the ***Einstein Equations***.

The Einstein Equations also predict that accelerating masses produce ***gravitational waves***, perturbations in the ***metric tensor*** of $\mathcal{M}$.

Black-Hole merger depictions (shamelessly stolen from LIGO website):





LIGO

LIGO (Laser Interferometer Gravitational-wave Observatory) is one of several recently constructed gravitational detectors.

The design of LIGO is based on measuring distance changes between objects in perpendicular directions as the ripple in the metric tensor propagates through the device.

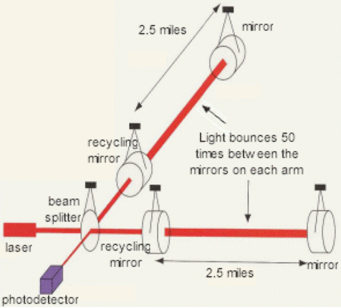
The two L-shaped LIGO observatories (in Washington and Louisiana), with legs at 1.5m meters by 4km, have phenomenal sensitivity, on the order of 10^{-15}m to 10^{-18}m.

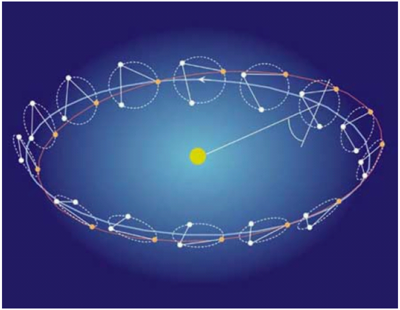
The LIGO arms in Louisiana and Hanford, Washington:

LIGO and LISA

LISA (Laser Interferometer Space Antenna) is different, space-based design currently under construction.

LISA is based on three triangulated L-shaped detectors mounted on satellites, separated by much larger distances than possible on Earth.





(Images courtesy of LIGO website)

LIGO Detection in Fall 2015

On 2-11-2016, the NSF announced an upcoming conference press for Feb 11 with the humble title “Scientists to Provide Update on the Search for Gravitational Waves”. It would not take place in some dusty lab, but rather at the National Press Club in Washington, DC.

When the press conference began on Feb. 11th, the LIGO Laboratory director David Reitze simply announced: “Ladies and gentlemen. We have detected gravitational waves. We did it.”

On 9-14-2015, both LIGO detectors nearly simultaneously registered a clear, loud, and violent inspiral, collision, merger, and ringdown of a binary black hole pair, each of which had a solar mass in range 10-50, with roughly equivalent of three solar masses in energy released as gravitational radiation. Radiation traveled outward from at speed of light, reaching LIGO detectors roughly 1.3 billion years later.

What was detected is an incredibly close match to computer simulations of wave emission from this type of binary collision, produced through very detailed numerical simulations of the full Einstein equations.

This talk focuses on the analysis and numerical treatment of some core PDE problems arising in gravitational wave science.

The Math: Curved Spacetime is a *Manifold*

The Einstein equations are a system of differential equations describing how spacetime curves in response to matter and energy.

Geometric piece of the equations can be understood by examining how derivatives in calculus must be modified when space (or spacetime) is curved:

$$\text{Flat:} \quad V^a{}_{,bc} - V^a{}_{,cb} = 0, \qquad V^a{}_{,b} = \frac{\partial V^a}{\partial x^b}.$$

$$\text{Curved:} \quad V^a{}_{;bc} - V^a{}_{;cb} = R^a{}_{dbc} V^d, \qquad V^a{}_{;b} = V^a{}_{,b} + \Gamma^a_{bc} V^c.$$

Let us note what form $R^a{}_{dbc}$ takes, and give names to some objects:

- $R^a{}_{dbc} = \Gamma^a_{bd,c} - \Gamma^a_{cd,b} + \Gamma^a_{ec}\Gamma^e_{bd} - \Gamma^a_{eb}\Gamma^e_{cd}$; Riemann tensor
- $R_{ab} = R_{acb}{}^c, \quad R = R_a{}^a$; Ricci tensor, scalar curvature
- $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$; Einstein tensor
- T_{ab}; Stress-energy tensor

The Einstein equations relate the mathematical object (G_{ab}) describing curvature of spacetime $\mathcal{M}$ to the mathematical object (T_{ab}) representing matter and energy content of our spacetime:

$$G_{ab} = \kappa T_{ab}, \qquad 0 \leqslant a \leqslant b \leqslant 3, \quad \kappa = 8\pi G/c^4. \quad \text{(10 equations)}$$

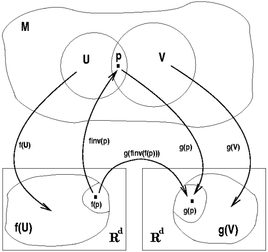
The Math: Curved Spacetime is a *Manifold*

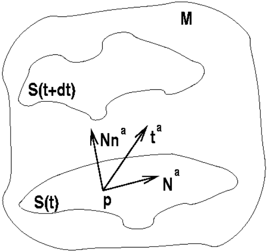
Space and time are all mixed up in Einstein's Equation: $G_{ab} = \kappa T_{ab}$.

It was hoped it could be reformulated as an ***initial-value problem***; "future" would then be determined by solution of a time-dependent differential equation for the "metric" of space at any future time.

This program was completed by various mathematicians by the 1950's; the famous book of Hawking & Ellis in 1973 summarizes this theory.

However, there remain a number of very important but difficult open problems in the mathematical theory of these equations, as well as in the development of reliable and efficient numerical methods.





Mathematics Research Problems in GR

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Some of the major open mathematical problems in GR concern a subset of the Einstein equations known as the ***Einstein constraint equations***.

These equations are non-dynamical, but they must hold at every moment of time for any solution to the Einstein equations; i.e., they must hold on every ***space-like hypersurface*** within the spacetime. The constraint equations have been studied intensively since the 1940's.

Mathematics research questions that arise with this problem are:

1. Do solutions to the constraints always exist? If so, are they unique?
2. How smooth are such solutions? Can we derive *a priori* bounds?
3. Can we develop a basic approximation theory for computing?
4. Can we establish error estimates for specific methods?
5. Can we design adaptive algorithms? Parallel algorithms?
6. Can we prove convergence of the algorithms?

Our research group has worked on various aspects of these questions over the last decade. In 2009 we contributed a new analysis approach proving existence results that led to a resurgence of activity on the problem.

Einstein Constraints and Conformal Method

Twelve-component Einstein evolution system for $(\hat{h}_{ab}, \hat{k}_{ab})$ on a foliation.

Constrained by coupled eqns on spacelike $\mathcal{M} = \mathcal{M}_t$, with $\hat{\tau} = \hat{k}_{ab}\hat{h}^{ab}$,

$$ {}^3\hat{R} + \hat{\tau}^2 - \hat{k}_{ab}\hat{k}^{ab} - 2\kappa\hat{\rho} = 0, \qquad \hat{\nabla}^a\hat{\tau} - \hat{\nabla}_b\hat{k}^{ab} - \kappa\hat{j}^a = 0. $$

York conformal decomposition: split initial data into 8 freely specifiable pieces plus 4 determined via: $\hat{h}_{ab} = \phi^4 h_{ab}$, $\hat{\tau} = \hat{k}_{ab}\hat{h}^{ab} = \tau$, and

$$ \hat{k}_{ab} = \phi^{-10}[\sigma^{ab} + (\mathcal{L}w)^{ab}] + \frac{1}{4}\phi^{-4}\tau h^{ab}, \quad \hat{j}^a = \phi^{-10} j^a, \quad \hat{\rho} = \phi^{-8}\rho. $$

Produces coupled elliptic system for conformal factor ϕ and a w^a:

$$ -8\Delta\phi + R\phi + \frac{2}{3}\tau^2\phi^5 - (\sigma_{ab} + (\mathcal{L}w)_{ab})(\sigma^{ab} + (\mathcal{L}w)^{ab})\phi^{-7} - 2\kappa\rho\phi^{-3} = 0, $$

$$ -\nabla_a(\mathcal{L}w)^{ab} + \frac{2}{3}\phi^6\nabla^b\tau + \kappa j^b = 0. $$

Differential structure on $\mathcal{M}$ defined through background 3-metric h_{ab}:

$$ (\mathcal{L}w)^{ab} = \nabla^a w^b + \nabla^b w^a - \frac{2}{3}(\nabla_c w^c)h^{ab}, \qquad \nabla_b V^a = V^a_{;b} = V^a_{,b} + \Gamma^a_{bc}V^c, $$

$$ V^a_{,b} = \frac{\partial V^a}{\partial x^b}, \qquad \Gamma^a_{bc} = \frac{1}{2}h^{ad}\left(\frac{\partial h_{db}}{\partial x^c} + \frac{\partial h_{dc}}{\partial x^b} - \frac{\partial h_{bc}}{\partial x^d}\right). \qquad (\Gamma^a_{bc} = \Gamma^a_{cb}) $$

The Conformal Method

Lichnerowicz and Choquet-Bruhat Papers: 1944 and 1958

- A. Lichnerowicz. L'integration des équations de la gravitation relativiste et le problème des n corps. *J. Math. Pures Appl.*, 23:37–63, 1944.
- Y. Choquet-Bruhat. Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires. *Acta Math.*, 88:141–225, 1952.

Some Key Conformal Method Papers: 1971–1996

- J. York. Gravitational degrees of freedom and the initial-value problem. *Phys. Rev. Lett.*, 26(26):1656–1658, 1971.
- J. Isenberg. Constant mean curvature solution of the Einstein constraint equations on closed manifold. Class. Quantum Grav. **12** (1995), 2249–2274.
- J. Isenberg and V. Moncrief. A set of nonconstant mean curvature solutions of the Einstein constraint equations on closed manifolds. Class. Quantum Grav. **13** (1996), 1819–1847.

No real progress made on "non-CMC case" during 1996–2008.

My Starting Point: 1998

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I was exposed to GR at Caltech as postdoc from 1993–1997.

Arriving at UCSD in 1998, I submitted my first NSF proposal for *Analysis and numerical treatment of the Einstein constraints*. *My Plan:*

1. Completely specify boundary-value formulation relevant to physicists.
2. Nail down what is known about the solution theory (existence, etc).
3. Nail down the literature in *a priori* estimates for the solution.
4. Use (1)-(3) to do numerical analysis research.

The numerical analysis research was going to be:

1. *A priori* error estimates for some choice of FEM.
2. Some type of solution theory for the discrete FEM system.
3. *A posteriori* error estimates (an indicator) for FEM.
4. Design of adaptive algorithms around the error indicator.
5. Design of nonlinear solver (based around Newton's method).
6. Design of fast linear solvers (multigrid and domain decomposition).
7. Parallel algorithms (got to have those).
8. Publish papers on all of the steps.

I ended up working on both lists (1998–2019); this is my talk.

The Conformal Method as an Elliptic System

Let $\mathcal{M}$ be a space-like Riemannian 3-manifold with (possibly empty) boundary submanifold $\partial\mathcal{M}$, split into disjoint submanifolds satisfying:

$$\partial_D\mathcal{M} \cup \partial_N\mathcal{M} = \partial\mathcal{M}, \qquad \partial_D\mathcal{M} \cap \partial_N\mathcal{M} = \emptyset. \qquad (\overline{\partial_D\mathcal{M}} \cap \overline{\partial_N\mathcal{M}} = \emptyset)$$

Metric h_{ab} associated with $\mathcal{M}$ induces boundary metric σ_{ab}, giving boundary value formulation of conformal method for ϕ and w^a:

$$\begin{aligned}
L\phi + F(\phi, w) &= 0, \ \text{in } \mathcal{M}, && \text{(Hamiltonian)} \\
\mathbb{L}w + \mathbb{F}(\phi) &= 0, \ \text{in } \mathcal{M}, && \text{(Momentum)} \\
(\mathcal{L}w)^{ab}\nu_b + C^a{}_b w^b &= V^a_\phi \text{ on } \partial_N\mathcal{M}, \quad \textit{and} \quad && w^a = w^a_D \text{ on } \partial_D\mathcal{M}, \\
(\nabla^a\phi)\nu_a + k_w(\phi) &= g \text{ on } \partial_N\mathcal{M}, \quad \textit{and} \quad && \phi = \phi_D \text{ on } \partial_D\mathcal{M},
\end{aligned}$$

where:

$$L\phi = -\Delta\phi, \quad (\mathbb{L}w)^a = -\nabla_b(\mathcal{L}w)^{ab},$$

$$F(\phi, w) = a_R\phi + a_\tau\phi^5 - a_w\phi^{-7} - a_\rho\phi^{-3}, \quad \mathbb{F}(\phi) = b^b_\tau\phi^6 + b^b_j,$$

with:

$$a_R = \frac{R}{8}, \ a_\tau = \frac{\tau^2}{12}, \ a_w = \frac{1}{8}[\sigma_{ab} + (\mathcal{L}w)_{ab}]^2, \ a_\rho = \frac{\kappa\rho}{4}, \ b^b_\tau = \frac{2}{3}\nabla^b\tau, \ b^b_j = \kappa j^b,$$

$$(\mathcal{L}w)^{ab} = \nabla^a w^b + \nabla^b w^a - \frac{2}{3}(\nabla_c w^c)h^{ab}, \qquad \nabla_b V^a = V^a{}_{;b} = V^a{}_{,b} + \Gamma^a_{bc}V^c,$$

$$V^a{}_{,b} = \frac{\partial V^a}{\partial x^b}, \qquad \Gamma^a_{bc} = \frac{1}{2}h^{ad}\left(\frac{\partial h_{db}}{\partial x^c} + \frac{\partial h_{dc}}{\partial x^b} - \frac{\partial h_{bc}}{\partial x^d}\right). \quad (\Gamma^a_{bc} = \Gamma^a_{cb})$$

The 1973–1995 CMC Results

$\nabla^b \tau = 0$: Constant Mean Curvature (CMC): $\Rightarrow$ constraints de-couple.

There were a number of CMC results generated during 1973–1995 by exploiting the fact that the constraint equations decouple.

You can solve the momentum constraint equation once and for all, and then you solve the Hamiltonian constraint once.

The research came down to understanding under what conditions the Hamiltonian constraint was solvable.

Some Key CMC Papers: 1974–1995

- N. Ó. Murchadha and J. York. Initial-value problem of general relativity I. General formulation and physical interpretation. *Phys. Rev. D*, 10(2):428–436, 1974.
- N. Ó. Murchadha and J. York. Initial-value problem of general relativity II. Stability of solution of the initial-value equations. *Phys. Rev. D*, 10(2):437–446, 1974.
- J. Isenberg. Constant mean curvature solution of the Einstein constraint equations on closed manifold. Class. Quantum Grav. **12** (1995), 2249–2274.

The 1996–2007 Near-CMC Results

$\nabla^b \tau \neq 0$: Non-CMC case: $\Rightarrow$ constraints couple.

In the Non-CMC case, the constraints couple together; through 1996 there were no results, until the Isenberg-Moncrief paper of 1996 under *near-CMC conditions* (to be explained). This led to several results.

Some of the Near-CMC Papers: 1996–2007

- J. Isenberg and V. Moncrief, *A set of nonconstant mean curvature solution of the Einstein constraint equations on closed manifolds*, Class. Quantum Grav. **13** (1996), 1819–1847.
- J. Isenberg and J. Park. Asymptotically hyperbolic non-constant mean curvature solutions of the Einstein constraint equations. *Class. Quantum Grav.*, 14:A189–A201, 1997.
- Y. Choquet-Bruhat, J. Isenberg, and J. York. Einstein constraint on asymptotically Euclidean manifolds. *Phys. Rev. D*, 61:084034, 2000.
- P. Allen, A. Clausen, and J. Isenberg. Near-constant mean curvature solutions of the Einstein constraint equations with non-negative Yamabe metrics. Available as arXiv:0710.0725 [gr-qc], 2007.

Yamabe Classification of Manifolds

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Take a smooth metric h_{ab} and a smooth positive function $u > 0$, and form a new metric through multiplication $u^4 h_{ab}$. A simple calculation shows that the scalar curvature R that h_{ab} generates, and the scalar curvature R_u that $u^4 h_{ab}$ generates, are related by the *Yamabe problem*:

$$-8\Delta u + Ru = R_u u^5.$$

If one has two metrics and they can be related through the Yamabe problem in this way, we say they are in the same *conformal class*.

Yamabe Classification of Smooth Metrics: Let $u > 0$ solve the Yamabe problem. Then h_{ab} lies in one of three disjoint conformal classes:

$$R_u > 0 \Rightarrow h_{ab} \in \mathcal{Y}^+, \quad R_u < 0 \Rightarrow h_{ab} \in \mathcal{Y}^-, \quad R_u = 0 \Rightarrow h_{ab} \in \mathcal{Y}^0.$$

Yamabe Classification of Non-Smooth Metrics: Yamabe problem on closed manifolds for rough metrics is still open; however, one can still get the following [HNT09]. Let $(\mathcal{M}, h)$ be a smooth, closed, connected Riemannian manifold with dimension $n \geqslant 3$ and consider a metric $h \in W^{s,p}$, where we assume $sp > n$ and $s \geqslant 1$. Then two conformally equivalent rough metrics cannot have scalar curvatures with distinct signs.

A Look at the 1996 Near-CMC Result

Fixed-point arguments involve composition $G(\phi) = T(\phi, S(\phi))$, where:

1. Given ϕ, solve MC for w: $w = S(\phi)$
2. Given w, solve HC for ϕ: $\phi = T(\phi, w)$

Map $S : X \to \mathcal{R}(S) \subset Y$ is MC solution map;
Map $T : X \times \mathcal{R}(S) \to X$ is some fixed-point map for HC.

Theorem: *(Isenberg-Moncrief) For case $R = -1$ on a closed manifold ($h_{ab} \in \mathcal{Y}^-$), strong smoothness assumptions, and near-CMC conditions, Isenberg-Moncrief show this is a contraction in Hölder spaces:*

$$[\phi^{(k+1)}, w^{(k+1)}] = G([\phi^{(k)}, w^{(k)}]).$$

Proof Outline: *Maximum principles, barriers, Banach algebra properties, near-CMC condition, contraction-mapping argument.* □

Near-CMC condition: $\|\nabla\tau\|_r < C \inf_{\mathcal{M}} |\tau|$, where L^r norm depends on context, can be viewed as engineering fixed-point map G to be a contraction, and ensures coupling between the two equations is weak.

Appears in two distinct places: (1) Construction of the contraction G, and (2) Construction of the set U on which G is a contraction.

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First Results for Far-From-CMC

[HNT08] MH, G. Nagy, and G. Tsogtgerel, *Far-from-constant mean curvature solutions of Einstein's constraint equations with positive Yamabe metrics*, Phys. Rev. Lett. **100** (2008), no. 16, 161101.1–161101.4, Available as arXiv:0802.1031 [gr-qc].

[HNT09] MH, G. Nagy, and G. Tsogtgerel, *Rough solutions of the Einstein constraints on closed manifolds without near-CMC conditions*, Comm. Math. Phys. **288** (2009), no. 2, 547–613, Available as arXiv:0712.0798 [gr-qc].

$\nabla^b \tau = 0$: Constant Mean Curvature (CMC): $\Rightarrow$ constraints de-couple.

As noted earlier, in Non-CMC case, constraints couple together; first result in 1996 for the near-CMC case, leading to other near-CMC results through 2007.

In 2008, we introduled a new analysis framework [HNT08, HNT09] that removed all need for CMC or near-CMC conditions in order to prove existence of solutions. This is now called the far-from-CMC case, or simply the non-CMC case.

Results in [HNT08, HNT09, Max09] (and most results after 2008) are based on using different variations of G, and then showing that these variatons have appropriate compactness properties. This is combined with the construction of *global barriers*, and then using topological fixed-point arguments.

The 2008 Framework: Mappings S and T

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We outline the near-CMC-free fixed-point argument from [HNT09].

We first make precise the definitions of the maps S and T.

To deal with the non-trivial kernel that exists for L on closed manifolds, fix an arbitrary positive *shift* $s > 0$. Now write the constraints as

$$L_s\phi + F_s(\phi, w) = 0, \tag{1}$$

$$(\mathbb{L}w)^a + \mathbb{F}(\phi)^a = 0, \tag{2}$$

where $L_s : W^{2,p} \to L^p$ and $\mathbb{L} : W^{2,p} \to L^p$ are defined as

$$L_s\phi := [-\Delta + s]\phi, \qquad (\mathbb{L}w)^a := -\nabla_b(\mathcal{L}w)^{ab},$$

and where $F_s : [\phi_-, \phi_+] \times W^{2,p} \to L^p$ and $\mathbb{F} : [\phi_-, \phi_+] \to L^p$ are

$$F_s(\phi, w) := [a_R - s]\phi + a_\tau\phi^5 - a_w\phi^{-7} - a_\rho\phi^{-3}, \qquad \mathbb{F}(\phi)^a := b^a_\tau\phi^6 + b^a_j.$$

Introduce the operators $S : [\phi_-, \phi_+] \to W^{2,p}$ and $T : [\phi_-, \phi_+] \times W^{2,p} \to W^{2,p}$ as

$$S(\phi) := -\mathbb{L}^{-1}\mathbb{F}(\phi), \tag{3}$$

$$T(\phi, w) := -L_s^{-1}F_s(\phi, w). \tag{4}$$

Both maps are well-defined when $s > 0$ (L_s is invertible) and when there are no conformal Killing vectors ($\mathbb{L}$ is invertible).

Schauder Approach to get at Non-CMC

Alternatives to Contraction Mapping Theorem that are more topological:

Theorem 1 (Schauder Theorem)

Let X be a Banach space, and let $U \subset X$ be a non-empty, convex, closed, bounded subset. If $G : U \to U$ is a compact operator, then there exists a fixed-point $u \in U$ such that $u = G(u)$.

Here is a variation of Schauder tuned for the constraints.

Theorem 2 (Coupled Schauder Theorem)

Let X and Y be Banach spaces, and let Z be a Banach space with compact embedding $X \hookrightarrow Z$. Let $U \subset Z$ be non-empty, convex, closed, bounded, and let $S : U \to \mathcal{R}(S) \subset Y$ and $T : U \times \mathcal{R}(S) \to U \cap X$ be continuous maps. Then, there exist $w \in \mathcal{R}(S)$ and $\phi \in U \cap X$ such that

$$\phi = T(\phi, w) \quad \textit{and} \quad w = S(\phi). \tag{5}$$

Proof Outline: *Show $G(\phi) = i \circ T(\phi, S(\phi)) : U \subset Z \to U \subset Z$ is compact and then use Schauder, where $i : X \to Z$ is (compact) canonical injection.* □

Rough Metrics

[HNT09] MH, G. Nagy, and G. Tsogtgerel, *Rough solutions of the Einstein constraints on closed manifolds without near-CMC conditions*, Comm. Math. Phys. **288** (2009), no. 2, 547–613, Available as arXiv:0712.0798 [gr-qc].

[HT13] MH and G. Tsogtgerel, *The Lichnerowicz equation on compact manifolds with boundary*, Class. Quantum Grav., 30 (2013), pp. 1–31. Available as arXiv:1306.1801 [gr-qc].

[HMT18] MH, C. Meier, and G. Tsogtgerel, *Non-CMC solutions of the Einstein constraint equations on compact manifolds with apparent horizon boundaries*, Comm. Math. Phys. **357** (2018), no. 2, 467–517, Available as arXiv:1310.2302 [gr-qc].

[BH15] A. Behzadan and MH, *Rough solutions of the Einstein constraint equations on asymptotically flat manifolds without near-CMC conditions*, Submitted. Available as arXiv:1504.04661 [gr-qc].

Relevant to the study of the Einstein evolution equations is the existence of solutions to the constraint equations for weak or *rough* background metrics h_{ab}. Initial results were developed for the CMC case in [yCB04, Max05a, Max06].

Requires careful examination of multiplication properties of the spaces [BH15].

Non-CMC rough solution results for closed manifolds appear in [HNT09], compact manifolds with boundary in [HT13, HMT18], AE manifolds in [BH15].

Brief Look at a Non-CMC Theorem

One of the three main results for rough non-CMC solutions on compact manifolds in [HNT09] is contained in the theorem.

Theorem 3 (Non-CMC $W^{s,p}$ solutions)

Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional closed Riemannian manifold. Let $h_{ab} \in W^{s,p}$ admit no conformal Killing field and be in $\mathcal{Y}^+(\mathcal{M})$, where $p \in (1, \infty)$ and $s \in (1 + \frac{3}{p}, \infty)$ are given. Select q and e to satisfy:

- $\frac{1}{q} \in (0, 1) \cap (0, \frac{s-1}{3}) \cap [\frac{3-p}{3p}, \frac{3+p}{3p}]$,
- $e \in (1 + \frac{3}{q}, \infty) \cap [s - 1, s] \cap [\frac{3}{q} + s - \frac{3}{p} - 1, \frac{3}{q} + s - \frac{3}{p}]$.

Assume that the data satisfies:

- $\tau \in W^{e-1,q}$ *if* $e \geqslant 2$, *and* $\tau \in W^{1,z}$ *otherwise, with* $z = \frac{3q}{3+\max\{0,2-e\}q}$,
- $\sigma \in W^{e-1,q}$, *with* $\|\sigma^2\|_\infty$ *sufficiently small,*
- $\rho \in W^{s-2,p} \cap L^\infty_+ \setminus \{0\}$, *with* $\|\rho\|_\infty$ *sufficiently small,*
- $\boldsymbol{j} \in \boldsymbol{W}^{e-2,q}$, *with* $\|\boldsymbol{j}\|_{e-2,q}$ *sufficiently small.*

Then there exists $\phi \in W^{s,p}$ with $\phi > 0$ and $\boldsymbol{w} \in \boldsymbol{W}^{e,q}$ solving the constraints.

Really Rough Metrics

Einstein Equations
Einstein Equations
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Compact Case (2013–)
AE Case (2014–)
Multiplicity (2017–)
Algorithms
AFEM Convergence
FEEC with AFEM
FEEC for Evolution
References

[HM13] MH and C. Meier, *Generalized solutions to semilinear elliptic PDE with applications to the Lichnerowicz equation*, Acta Appicandae Mathematicae, 130 (2014), pp. 163–203. Available as arXiv:1112.0351 [math.NA].

One of the difficulties associated with obtaining rough solutions to the conformal formulation is that the spaces $W^{s,p}(\mathcal{M})$ are not closed under multiplication unless $s > d/p$ (where d is the spatial dimension).

This restriction is a by-product of a more general problem, which is that there is no well-behaved definition of distributional multiplication that allows for the multiplication of arbitrary distributions.

Limits spaces one considers when developing weak formulation of a given elliptic partial differential equation, and places a restriction on regularity of the specified data $(g_{ab}, \tau, \sigma, \rho, j)$ of the constraint equations.

In [HM13], we extend the work of Mitrovic-Pilipovic (2006) and Pilipovic-Scarpalezos (2006) to solve problems similar to Hamiltonian constraint with distributional coefficients in *Colombeau algebras*.

These generalized spaces allows one to circumvent the restrictions associated with Sobolev coefficients and data, and thereby consider problems with coefficients and data of much lower regularity.

Compact with Boundary Case

[HT13] MH and G. Tsogtgerel, *The Lichnerowicz equation on compact manifolds with boundary*, Class. Quantum Grav., 30 (2013), pp. 1–31. Available as arXiv:1306.1801 [gr-qc].

[HMT18] MH, C. Meier, and G. Tsogtgerel, *Non-CMC solutions of the Einstein constraint equations on compact manifolds with apparent horizon boundaries*, Comm. Math. Phys. **357** (2018), no. 2, 467–517, Available as arXiv:1310.2302 [gr-qc].

Compact manifolds $\mathcal{M}$ with boundary $\Sigma = \partial\mathcal{M}$ emerge when one eliminates asymptotic ends or singularities from the manifold.

For Lichnerowicz, one needs to impose boundary conditions for ϕ.

On asymptotically flat manifolds, one has [YP82]

$$\phi = 1 + Ar^{2-n} + \varepsilon, \qquad \text{with} \quad \varepsilon = O(r^{1-n}), \quad \text{and} \quad \partial_r\varepsilon = O(r^{-n}), \tag{6}$$

where A is multiple total energy, r is the flat-space radial coordinate. Idea is: cut out asymptotically Euclidean end along the sphere with large radius r and impose Dirichlet condition $\phi \equiv 1$ at boundary.

Improvement via differentiating (6) with respect to r and eliminating A:

$$\partial_r\phi + \frac{n-2}{r}(\phi - 1) = O(r^{-n}). \tag{7}$$

Equating right hand side to zero gives accurate total energy.

Approximating Black Hole Data

Main approach: excise region around singularities and solve in exterior.

Such are "inner"-boundaries; again need boundary conditions.

In 1982, York and Piran [YP82] introduce

$$\partial_r \phi + \frac{n-2}{2a}\phi = 0, \qquad \text{for } r = a. \tag{8}$$

Means $r = a$ is a minimal surface; under appropriate data conditions minimal surface is a *trapped surface*.

Trapped surface key: implies existence of event horizon outside surface.

Trapped surface conditions more general than minimal surface can be derived using expansion scalars for outgoing/ingoing future directed null geodesics (Dain 2004 [Dai04], Maxwell 2005 [Max05b]).

After some work, one sees Dain and Maxwell approaches both lead to inner and outer boundary conditions (our $k_w(\phi)$ earlier) of form:

$$\partial_\nu \phi + b_H \phi + b_\theta \phi^e + b_\tau \phi^{\bar{q}} + b_w \phi^{-\bar{q}} = 0, \quad \bar{q} = n/(n-2). \tag{9}$$

Minimal surface condition (8) corresponds to the choice $b_\theta = b_\tau = b_w = 0$, and $b_H = \frac{n-2}{2} H$, $H =$ mean extrinsic curvature of Σ.

The outer Robin condition (7) is $b_H = (n-2)H$, $b_\theta = -(n-2)H$ with $e = 0$, and $b_\tau = b_w = 0$.

Constraints Are Coupled Through Boundary

Something interesting here: the Lichnerowicz equation couples to the momentum constraint through the boundary conditions.

Even in CMC (constant mean curvature) setting (where $\tau \equiv \mathrm{const}$), constraint equations generally do not *decouple.*

The only reasonable way to decouple the constraints is to consider $\tau \equiv 0$ and $e = -\bar{q}$.

Main tools used in paper are order-preserving maps iteration together with maximum principles and some results from conformal geometry.

These techniques sensitive to signs of coefficients in (9).

Defocusing case (preferred signs): $(e-1)b_\theta \geqslant 0$, $b_\tau \geqslant 0$, and $b_w \leqslant 0$.

Non-Defocusing case: Otherwise.

Results for defocusing case (terminology motivated by dispersive equations) more or less complete (see below).

CMC Case: Main Results in [HT13]

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The main results and supporting tools appearing in [HT13] are:

- Justification of Yamabe classification of rough metrics on compact manifolds with boundary.
- Basic result on conformal invariance of Lichnerowicz equation.
- A uniqueness result for the Lichnerowicz equation.
- An order-preserving maps theorem for manifolds with boundary.
- Construction of upper and lower barriers that respect the trapped surface conditions.
- Combination of the results above to produce a fairly complete existence and uniqueness theory for the defocusing case.
- Combination of the results above to produce some partial results for the non-defocusing case.
- Some perturbation results (looking ahead to the asymptotically Euclidean case).
- Existence results for Lichnerowicz equation (not full CMC case).

Some results for smooth metrics, ignoring the boundary coupling, appear in [Dilt14].

Non-CMC case: Main Results in [HMT18]

What about the non-CMC case?

In fact, even the CMC case was not yet discussed; this is because the CMC assumption does not actually decouple the constraints due to the boundary coupling, and we have only solved the Lichnerowicz equation.

The extension of the results in [HT13] to the non-CMC (far, near, and also CMC itself) is considered in [HMT18].

Some of the main results appearing in [HMT18] are:

- Number of necessary supporting results for momentum constraint that were not needed for pure Lichnerowicz case in [HT13].
- Construction of upper and lower barriers that respect trapped surface conditions in coupled setting (delicate boundary coupling).
- Combination of Schauder argument from [HNT09] with results for Lichnerowicz equation from [HT13] to give existence results for near-CMC and far-CMC data, analogous to known results for closed manifolds.
- CMC case comes as (still coupled) special case of near-CMC result.

Yamabe Classes: Rough/Compact/Boundary

Yamabe classification of rough metrics: The Yamabe problem for rough metrics on compact manifolds with boundary is again still open; the work [Esc92, Esc96] was for smooth metrics. However, as in the closed case, one can still get the following result [HT13] which is all we need:

Theorem 4 (Yamabe Classification of Rough Metrics)

Let (M, g) be a smooth, compact, connected Riemannian manifold with boundary, where we assume that the components of the metric g are (locally) in $W^{s,p}$, with $sp > n$ and $s \geqslant 1$. Let the dimension of M be $n \geqslant 3$. Then, the following are equivalent:

a) $\mathcal{Y}_g > 0$ ($\mathcal{Y}_g = 0$ or $\mathcal{Y}_g < 0$).

b) $\mathcal{Y}_g(q, r, b) > 0$ (resp. $\mathcal{Y}_g(q, r, b) = 0$ or $\mathcal{Y}_g(q, r, b) < 0$) for any $q \in [2, 2\bar{q})$, $r \in [2, \bar{q} + 1)$ with $q > r$, and any $b \in \mathbb{R}$.

c) *There is a metric in $[g]$ whose scalar curvature is continuous and positive (resp. zero or negative), and boundary mean curvature is continuous and has any given sign (resp. is identically zero, has any given sign).*

In particular, two conformally equivalent metrics cannot have scalar curvatures with distinct signs.

Asymptotically Euclidean Case

Some Research Problems in GR

Michael Holst

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Compact Case (2013–)
AE Case (2014–)
Multiplicity (2017–)

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FEEC for Evolution

References

[DIM14] J. Dilts, J. Isenberg, R. Mazzeo, and C. Meier. Non-cmc solutions of the Einstein constraint equations on asymptotically euclidean manifolds. 2014 Class. Quantum Grav. **31** (2014), 065001.

[HMa14] MH and C. Meier, *Non-CMC solutions of the Einstein constraint equations on asymptotically Euclidean manifolds with apparent horizon boundaries*, Class. Quantum Grav., 32 (2014), No. 2, pp. 1-25. Available as arXiv:1403.4549 [gr-qc].

[BH15] A. Behzadan and MH, *Rough solutions of the Einstein constraint equations on asymptotically flat manifolds without near-CMC conditions*, Submitted. Available as arXiv:1504.04661 [gr-qc].

The most complete mathematical model of general relativity involves the evolution and constraint equations on open, asymptotically Euclidean manifolds, with black hole interior boundary conditions.

Existence results analogues to those for closed manifolds have been known since shortly after the closed results developed.

In [HMa14], we develop non-CMC existence results for asymptotically Euclidean manifolds with black hole interior boundaries.

In [BH15], we extend this work to rough metrics as part of the research program begun in 2004/2005.

Multiplicity Results

[HK11] MH and V. Kungurtsev, *Numerical bifurcation analysis of conformal formulations of the Einstein constraints*, Phys. Rev. D, 84 (2011), pp. 124038(1)–124038(8). Available as arXiv:1107.0262 [math.NA].

[HM14] MH and C. Meier, *Non-uniqueness of solutions to the conformal formulation*, Available as arXiv:1210.2156 [gr-qc].

[DH17] J. Dilts, MH, T. Kozareva, and D. Maxwell, *Numerical Bifurcation Analysis of the Conformal Method*, Submitted. Available as arXiv:1710.03201 [gr-qc].

[HMM18] MH, D. Maxwell, and R. Mazzeo, *Conformal Fields and the Structure of the Space of Solutions of the Einstein Constraint Equations*, Submitted. Available as arXiv:1711.01042 [gr-qc].

New non-CMC existence results lack uniqueness. In 2009, Maxwell explicitly demonstrated existence of multiple solutions for a special symmetric model [Max09b], and now also [Max14b].

More general multiplicity result now apparently shown by [GiNg15b].

Folds in solution curves observed numerically by Pfeiffer, O'Murchadha and others for non-standard formulations of the constraints; i.e., the mechanism is different from Maxwell and Gicquaud-Nguyen results.

Multiplicity Results

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In [HK11], we applied pseudo-arclength numerical continuation to numerically track the parameterized solution curve in the problem previously examined by Pfeiffer and O'Murchadha, and numerically identify a fold.

In [HM14] we use Liapunov-Schmidt reduction from analytic bifurcation theory to show that linearization of the non-standard system develops a one-dimensional kernel; related results appear in [ChGi15].

Both papers employ the following λ-parameterization of the model:

$$L\phi + a_R\phi + \lambda^2 a_\tau \phi^5 - a_w\phi^{-7} - e^{-\lambda} a_\rho \phi^5 = 0, \qquad (10)$$

$$\mathbb{L}w + \lambda b_\tau^b \phi^6 = 0. \qquad (11)$$

In [DH17] we perform careful numerical bifurcation analysis of standard formulation of the conformal method, and demonstrate complex fold and bifurcation phenomena consistent with analytical results.

In [HMM18] we examine the drift method, a new alternative to the conformal method, and show it reproduces conformal method results. Unlike conformal method, it can be applied even when the underlying metric admits conformal Killing (but not true Killing) vector fields. We also prove that the absence of true Killing fields is generic.

Filling holes after we have fallen into them...

[BH15] A. Behzadan and MH, *Multiplication in Sobolev-Slobodeckij spaces, revisited*. Submitted for publication. Available as arXiv:1512.07379 [math.AP].

[BH18b] A. Behzadan and MH, *Sobolev-Slobodeckij spaces on compact manifolds, revisited*. Submitted for publication. Available as arXiv:1704.07930v3 [math.AP].

[BH17] A. Behzadan and MH, *On certain geometric operators between Sobolev spaces of sections of tensor bundles on compact manifolds equipped with rough metrics*. Submitted for publication. Available as arXiv:1704.07930v2 [math.AP].

[BH18a] A. Behzadan and MH, *Some remarks on the space of locally Sobolev-Slobodeckij functions*. Submitted for publication. Available as arXiv:1806.02188 [math.AP].

Trying to fill in some holes in the literature.

Outline (Starting Part 3)

1 The Einstein Equations
- Einstein Evolution and Constraint Equations
- The Conformal Method(s) of 1944, 1973, 1974
- Analysis of the Conformal Method: 1973–1995, 1996–2007
- The 1973–1995 CMC Results
- The 1996–2007 Near-CMC Results

2 New Conformal Method Results from our Group (2008-2019)
- First Results for Far-From-Constant Mean Curvature (2008–)
- Results for Rough Metrics (2009–)
- Compact with Boundary Case (2013–)
- Asymptotically Euclidean Case (2014–)
- Multiplicity through Analytic Bifurcation Theory (2017–)

3 Design and Analysis of Approximations and Algorithms
- AFEM Convergence with Goal Functions
- AFEM Convergence in the FEEC Setting
- FEEC A Priori Error Estimates for Evolution Problems

4 References

Design and Analysis of Approximations

[H01] MH, *Adaptive numerical treatment of elliptic systems on manifolds*, Adv. Comput. Math., 15 (2001), pp. 139–191. Available as arXiv:1001.1367 [math.NA].

[HS12a] MH and A. Stern, *Geometric variational crimes: Hilbert complexes, finite element exterior calculus, and problems on hypersurfaces*, Found. Comput. Math. **12** (2012), no. 3, 263–293, Available as arXiv:1005.4455 [math.NA].

[HS12b] MH and A. Stern, *Semilinear mixed problems on Hilbert complexes and their numerical approximation*, Found. Comput. Math. **12** (2012), no. 3, 363–387, Available as arXiv:1010.6127 [math.NA].

[GL19] MH and M. Licht, *Geometric transformation of finite element methods: Theory and applications*. Submitted for publication. Available as arXiv:1809.10354 [math.NA].

We consider now second-order elliptic on Riemanniann manifolds

$$-\Delta u + f(u) = 0,$$

coupled to other equations, perhaps involving the vector Laplacean:

$$-\Delta u = -\operatorname{grad}\ \operatorname{div} u + \operatorname{curl}\ \operatorname{curl} u.$$

Our goal is to develop *good PG methods* for these problems.

Mixed Formulations and Stable Methods

Consider the vector Laplacean:

$$-\Delta u = -\,\text{grad}\;\text{div}\;u + \text{curl}\;\text{curl}\;u,$$

and natural variational formulation: Find $u \in H(\text{curl};\Omega) \cap H_0(\text{div};\Omega)$ s.t.

$$\int_\Omega [(\nabla\cdot u)(\nabla\cdot v)+(\nabla\times u)\cdot(\nabla\times v)]\,dx = \int_\Omega f\cdot v\,dx,\;\;\forall v \in H(\text{curl};\Omega)\cap H_0(\text{div};\Omega).$$

Mixed formulation is an alternative: Find $(\sigma, u) \in H^1(\Omega) \times H(curl;\Omega)$ s.t.

$$\int_\Omega (\sigma\tau - u\cdot\nabla\tau)\,dx = 0, \quad \forall \tau \in H^1(\Omega),$$

$$\int_\Omega [\nabla\sigma\cdot v + (\nabla\times u)\cdot(\nabla\times v)]\,dx = \int_\Omega f\cdot v\,dx, \quad \forall v \in H(curl;\Omega).$$

FEM based on FIRST formulation: Does not correctly capture either geometry (domains with corners) or topology (non-simply connected domains); see worked examples in the core FEEC papers [AFW06, AFW10].

FEM based on SECOND formulation: Turns out to work extremely well.

Why is this the case?

Helmholtz-Hodge Decomposition

Given any vector field $f \in L^2(\Omega)$, we have the *Helmholtz-Hodge* orthogonal decomposition into curl-free, divergence-free, and harmonic functions:

$$f = \nabla p + \nabla \times q + h,$$

where h is harmonic (divergence- and curl-free).

The mixed formulation is essentially computing this decomposition for $h = 0$, and finite element methods are somehow exploiting this.

Connection to *de Rham cohomology*: The space of harmonic forms is isomorphic to the first *de Rham cohomology* of the domain Ω; the number of holes in Ω is the first Betti number, and creates obstacles to well-posed formulations of elliptic problems.

Q: What is an appropriate mathematical framework for understanding this abstractly, that will allow for a methodical construction of “good” finite element methods for these types of problems?

A: Hilbert (Banach, Bochner, etc) Complexes.

Hilbert Complexes

Hilbert complex (W, d) consists of sequence of Hilbert spaces W^k with closed densely-defined linear maps $\mathrm{d}^k : V^k \subset W^k \to V^{k+1} \subset W^{k+1}$ s.t. $\mathrm{d}^k \circ \mathrm{d}^{k-1} = 0$ for each k.

$$\cdots \longrightarrow V^{k-1} \xrightarrow{\mathrm{d}^{k-1}} V^k \xrightarrow{\mathrm{d}^k} V^{k+1} \longrightarrow \cdots$$

Given Hilbert complex (W, d), the *domain complex* (V, d) consists of the domains $V^k \subset W^k$, endowed with the graph inner product

$$\langle u, v \rangle_{V^k} = \langle u, v \rangle_{W^k} + \langle \mathrm{d}^k u, \mathrm{d}^k v \rangle_{W^{k+1}}.$$

A canonical example is the L^2-de Rham complex of differential forms on a Riemannian n-manifold Ω:

$$0 \longrightarrow H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}, \Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow 0$$

Special case is $\Omega \subset \mathbb{R}^3$, where Ω is a two-dimensional hypersurface.

Hodge Decomposition and Poincaré Inequality

Given Hilbert complex (W, d):

- space of *k-cocycles*: $\mathfrak{Z}^k = \ker \mathrm{d}^k$
- space of *k-coboundaries*: $\mathfrak{B}^k = \mathrm{d}^{k-1} V^{k-1}$
- dual space of *k-coboundaries*: $\mathfrak{B}^*_k = \mathrm{d}^*_k V^*_{k+1}$
- *kth harmonic space* is intersection: $\mathfrak{H}^k = \mathfrak{Z}^k \cap \mathfrak{B}^{k\perp}$
- *kth cohomology space* is quotient: $\mathfrak{Z}^k / \mathfrak{B}^k$
- *kth reduced cohomology space* is quotient: $\mathfrak{Z}^k / \overline{\mathfrak{B}^k}$.

One says (W, d) is *bounded* if each d^k is bounded, *closed* if each d^k has closed range.

Harmonic space $\mathfrak{H}^k$ is isomorphic to reduced cohomology space $\mathfrak{Z}^k / \overline{\mathfrak{B}^k}$.

Closed complex: reduced cohomology space $\equiv$ cohomology space.

If (W, d) is closed, we have abstract versions of *Hodge decomposition*

$$W^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{B}^*_k,$$

$$V^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp}.$$

and *Poincaré inequality* ($d^k : \mathfrak{Z}^{k\perp} \to \mathfrak{B}^{k+1}$ is a V-bounded bijection)

$$\|v\|_V \leqslant C_p \|d^k v\|_V, \quad \forall \mathfrak{Z}^{k\perp}.$$

Mixed Variational Formulation

Some Research Problems in GR

Michael Holst

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Abstract "mixed version" of Poisson equation $-\Delta u = f$ for scalar functions via abstract Hodge Laplacian for Hilbert complexes.

Abstract Hodge Laplacian is operator $L = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$, unbounded $W^k \to W^k$ with domain

$$D_L = \left\{ u \in V^k \cap V_k^* \;\middle|\; \mathrm{d}u \in V_{k+1}^*,\ \mathrm{d}^* u \in V^{k-1} \right\}.$$

If $u \in D_L$ solves $Lu = f$, then satisfies variational principle

$$\langle \mathrm{d}u, \mathrm{d}v \rangle + \langle \mathrm{d}^* u, \mathrm{d}^* v \rangle = \langle f, v \rangle, \quad \forall v \in V^k \cap V_k^*.$$

Problems: well-posedness (harmonic functions in kernel) and building (finite element) subspaces.

Solution: mixed problem: Find $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ s.t.

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle u, \mathrm{d}\tau \rangle &= 0, && \forall \tau \in V^{k-1}, \\ \langle \mathrm{d}\sigma, v \rangle + \langle \mathrm{d}u, \mathrm{d}v \rangle + \langle p, v \rangle &= \langle f, v \rangle, && \forall v \in V^k, \\ \langle u, q \rangle &= 0, && \forall q \in \mathfrak{H}^k. \end{aligned} \tag{12}$$

First equation: implies $\sigma = \mathrm{d}^* u$, weakly enforces condition $u \in V^k \cap V_k^*$.

Second equation: $\langle p, v \rangle$ allows for existence of solutions when $f \not\perp \mathfrak{H}^k$.

Third equation fixes issue of non-uniqueness by requiring $u \perp \mathfrak{H}^k$.

PG in Hilbert Complexes: AFW and FEEC

Galerkin-like method: Find $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ s.t.

$$\begin{aligned} \langle \sigma_h, \tau \rangle - \langle u_h, \mathrm{d}\tau \rangle &= 0, && \forall \tau \in V_h^{k-1}, \\ \langle \mathrm{d}\sigma_h, v \rangle + \langle \mathrm{d}u_h, \mathrm{d}v \rangle + \langle p_h, v \rangle &= \langle f, v \rangle, && \forall v \in V_h^k, \\ \langle u_h, q \rangle &= 0, && \forall q \in \mathfrak{H}_h^k. \end{aligned} \tag{13}$$

One of the main results in the core papers on the Finite Element Exterior Calculus (FEEC) [AFW06, AFW10] is the following.

Let (V_h, d) be a family of subcomplexes of the domain complex (V, d) of a closed Hilbert complex, parameterized by h and admitting uniformly V-bounded cochain projections π_h, and let $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ be the solution of (12) and $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ the solution of problem (13). Then

$$\begin{aligned} &\|\sigma - \sigma_h\|_V + \|u - u_h\|_V + \|p - p_h\| \\ \leq\; & C\Big(\inf_{\tau \in V_h^{k-1}} \|\sigma - \tau\|_V + \inf_{v \in V_h^k} \|u - v\|_V + \inf_{q \in V_h^k} \|p - q\|_V + \mu \inf_{v \in V_h^k} \|P_{\mathfrak{B}} u - v\|_V \Big), \end{aligned}$$

where $\mu = \mu_h^k = \sup_{\substack{r \in \mathfrak{H}^k \\ \|r\|=1}} \left\| \left(I - \pi_h^k \right) r \right\|$.

Variational Crimes

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Basic "variational" problem: Find $u \in V$ such that

$$B(u, v) = F(v), \quad \forall v \in V, \tag{14}$$

where V is a Hilbert space, $B\colon V \times V \to \mathbb{R}$ is a bounded, coercive bilinear form, and $F \in V^*$ is a bounded linear functional.

Galerkin method: Find $u_h \in V_h \subset V$ such that

$$B(u_h, v) = F(v), \quad \forall v \in V_h. \tag{15}$$

Problem: One often cannot compute V_h, and/or bilinear form $B(\cdot, \cdot)$ and functional $F(\cdot)$ on a subspace of V.

Left with approximating space $V_h \not\subset V$, approximate forms $B_h\colon V_h \times V_h \to \mathbb{R}$ and $F_h \in V_h^*$.

Generalized Galerkin: Find $u_h \in V_h$ such that

$$B_h(u_h, v) = F_h(v), \quad \forall v \in V_h \not\subset V. \tag{16}$$

Called "variational crimes". Framework for the analysis: *Strang lemmas*.

Variational Crimes in Hilbert Complexes

Approximation by an arbitrary complex:

$$
\begin{array}{ccccccc}
\cdots & \longrightarrow & V^k & \xrightarrow{\mathrm{d}^k} & V^{k+1} & \longrightarrow & \cdots \\
 & & i_h^k \uparrow \downarrow \pi_h^k & & i_h^{k+1} \uparrow \downarrow \pi_h^{k+1} & & \\
\cdots & \longrightarrow & V_h^k & \xrightarrow{\mathrm{d}_h^k} & V_h^{k+1} & \longrightarrow & \cdots .
\end{array}
$$

FEEC considers $W_h \subset W$ a subcomplex, i_h inclusion of W_h into W.

Our new approximation problem: Find $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ s.t.

$$
\begin{aligned}
\langle \sigma_h, \tau_h \rangle_h - \langle u_h, \mathrm{d}_h \tau_h \rangle_h &= 0, & \forall \tau_h &\in V_h^{k-1}, \\
\langle \mathrm{d}_h \sigma_h, v_h \rangle_h + \langle \mathrm{d}_h u_h, \mathrm{d}_h v_h \rangle_h + \langle p_h, v_h \rangle_h &= \langle f_h, v_h \rangle_h, & \forall v_h &\in V_h^k, \\
\langle u_h, q_h \rangle_h &= 0, & \forall q_h &\in \mathfrak{H}_h^k.
\end{aligned}
\tag{17}
$$

In [HS12a] we show that: If $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ is a solution to (12) and $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ is a solution to (17), then

$$
\begin{aligned}
&\|\sigma - i_h\sigma_h\|_V + \|u - i_h u_h\|_V + \|p - i_h p_h\| \\
\leq C\Big(&\inf_{\tau \in i_h V_h^{k-1}} \|\sigma - \tau\|_V + \inf_{v \in i_h V_h^k} \|u - v\|_V + \inf_{q \in i_h V_h^k} \|p - q\|_V + \mu \inf_{v \in i_h V_h^k} \|P_{\mathfrak{B}} u - v\|_V \\
&+ \|f_h - i_h^* f\|_h + \|I - J_h\| \, \|f\| \Big).
\end{aligned}
$$

Application to Surface Finite Elements

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Here, $\mu = \mu_h^k = \sup_{\substack{r \in \mathfrak{H}^k \\ \|r\|=1}} \left\| \left(I - i_h^k \pi_h^k \right) r \right\|$.

Theorem establishes convergence, as long as our approximations satisfy $\|I - J_h\| \to 0$ and $\|f_h - i_h^* f\|_h \to 0$ when $h \to 0$. We show $f_h = \pi_h f$ is sufficient to get a convergent solution.

In the case of surface finite element methods (SFEM), the term involving J_h is determined by the approximation order of the approximate surface.

In particular, we show how *a priori* error estimates of Dziuk (1988) and Demlow (2009) for SFEM for the Laplace Beltrami on 2- and 3-surfaces can be completely recovered by our theorem.

However, our FEEC approach gives more transparent error analysis for SFEM, looking more like Strang-type lemmas in Hilbert complexes.

Moreover, our results hold for general dimensional Euclidean hypersurfaces of co-dimension one, and general Riemannian manifolds as well as more general mixed formulations involving general k-forms.

Results extended in [HS12b] to **semilinear case:** One needs only local Lipschitz result (again) together with some nonlinear analysis.

Goal-Oriented AFEM Convergence Results

[CHX07] L. Chen, MH, and J. Xu, *The finite element approximation of the nonlinear Poisson-Boltzmann Equation*, SIAM J. Numer. Anal., 45 (2007), pp. 2298–2320. Available as arXiv:1001.1350 [math.NA].

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[HPZ15] MH, S. Pollock, and Y. Zhu, *Convergence of goal-oriented adaptive finite element methods for semilinear problems*, Computing and Visualization in Science, 17 (2015), pp. 43–63. Available as arXiv:1203.1381 [math.NA].

In [HP16, HPZ15] we develop AFEM convergence theory for a class of goal-oriented adaptive finite element algorithms (GOAFEM).

Following Mommer and Stevenson (2009) for symmetric problems, in [HP16] we establish GOAFEM contraction for nonsymmetric problems. Our approach uses newer contraction frameworks.

In [HPZ15], we prove convergence of GOAFEM for semilinear problems. We first establish quasi-error contraction of primal problem, then establish contraction of combined primal-dual quasi-error, giving convergence with respect to the quantity of interest.

An FEEC-AFEM Convergence Result

[CHX09] L. Chen, MH, and J. Xu, *Convergence and optimality of adaptive mixed finite element methods*, Math. Comp., 78 (2009), pp. 35–53. Available as arXiv:1001.1353 [math.NA].

[HLM19] MH, Y. Li, A. Mihalik, and R. Szypowski, *Convergence and optimality of adaptive mixed methods for Poisson's equation in the FEEC framework*. Accepted for publication in J. Comput. Math., 2019. Available as arXiv:1306.1886 [math.NA].

[HLL19] MH, Y. Li, and M. Licht, *Convergence and optimality of adaptive mixed methods on surfaces in the FEEC framework*. In Preparation, 2019.

AFEM Convergence theory for mixed methods not complete; main difficulty is lack of error (quasi-)orthogonality.

In [CHX09], we establish convergence and optimality of AFEM for mixed Poisson on simply connected domains in two dimensions.

Argument: quasi-orthogonality result exploits error orthogonal to divergence free subspace, with non-divergence-free part bounded by data oscillation via discrete stability.

In 2012, Demlow and Hirani develop FEEC a posteriori indicators.

In [HLM19] we prove AFEM-FEEC convergence complexity for Hodge-Laplace (k=n) on domains of arbitrary topology and dimension.

FEEC Estimates for Evolution Problems

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[GH19] E. Gawlik and MH, *Finite element exterior calculus for parabolic problems on evolving surfaces*. In Preparation, 2019.

[GHL19] E. Gawlik, MH, and M. Licht, *A Scott-Zhang interpolant and piecewise Bramble-Hilbert lemma for finite element exterior calculus*. In Preparation, 2019.

In [GHZ17], we extend most of the results in [HS12a, HS12b] to mixed formulations of linear and semilinear parabolic and hyperbolic problems.

Combines FEEC for elliptic problems with classical approaches to evolution problems via semi-discrete FEM, viewing solutions as lying in time-parameterized Banach spaces (*Bochner* spaces).

Building on Thomée (2006) (and others) for parabolic problems and Geveci (1988) (and others) for hyperbolic problems, establish *a priori* estimates for Galerkin approximation in natural Bochner norms.

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In [GHZ17] we recover results of Thomée and Geveci for two-dimensional domains and lowest-order mixed method as a special case, giving extension to arbitrary spatial dimension and entire family of mixed methods.

We also show how the Holst and Stern framework [HS12a, HS12b] allows for extensions of these results to semi-linear evolution problems.

In [HT18], the results for parabolic problems in [GHZ17] are extended to (fixed) Riemannian hypersurfaces.

In [GH19], the results for parabolic problems on hypersurfaces in [HT18] are extended to *evolving* Riemannian hypersurfaces.

Some key supporting results needed for the results in [GH19] appear in [GHL19].

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Thank You for Listening!

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Some Research Problems in GR

Michael Holst

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