SOME BOUNDS ON THE NUMBER OF DETERMINING NODES FOR WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

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Abstract. We derive new upper bounds on the dimension of the determining set for weak solutions of the Navier-Stokes equations. Our results extend the recent bounds due to Jones and Titi in three ways. First, the bounds are derived under the minimal H^1-regularity required to define a weak solution of the Navier-Stokes equations. Second, the new bounds are valid for arbitrary polyhedral domains, whereas previous results were derived for the unit square. Third, our results hold also in the three-dimensional case, whereas previous bounds were restricted to two dimensions. The generalizations are made possible through the use of some new results in polynomial approximation theory of non-smooth functions in Sobolev spaces.

Key Words. Approximation theory, Sobolev spaces, regularity, weak solutions, Navier-Stokes equations, determining nodes, Grashof number, Gronwall lemma.

1. Introduction. Consider a viscous incompressible fluid in $\Omega \subset \mathbb{R}^d$, where Ω is a bounded open domain with Lipshitz continuous boundary, and where $d = 2$ or $d = 3$. Given the kinematic viscosity $\nu > 0$, and the scalar volume force function $f(x,t)$ for each $x \in \Omega$ and $t \in (0,\infty)$, the governing Navier-Stokes equations for the fluid velocity vector $u = u(x,t)$ and the scalar pressure $p = p(x,t)$ are:

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f \quad \text{in} \quad \Omega \times (0,\infty), \tag{1}$$

$$\nabla \cdot u = 0 \quad \text{in} \quad \Omega \times (0,\infty). \tag{2}$$

Also provided are initial conditions $u(0) = u_0$, as well as appropriate boundary conditions on $\partial\Omega$. The no-slip boundary case is considered in this paper: $u(x,t) = 0$, $\forall x \in \partial\Omega$, $t \in (0,\infty)$. The question of determining sets (nodes, modes, or volumes) for strong (that is, H^2-regular) solutions of the Navier-Stokes equations, for both no-slip and periodic boundary conditions, has been considered recently in [5, 6, 8]. In particular, we mention the extensive recent work of Jones and Titi on the subject [9, 10]. The relevance of the determining set question in theoretical fluid mechanics is well-established; see [10] for an excellent discussion.

Recall that H^2-regular functions have point-wise values in both two and three dimensions (since the imbedding $H^2 \hookrightarrow C^0$ holds in both cases), so that a set of determining nodes may be defined as follows.

Definition 1. *Assume that $u(t)$ solves (1)–(2) with source function $f(t)$, and that $v(t)$ solves (1)–(2) with source function $g(t)$, where $\lim_{t\to\infty} \|f(t) - g(t)\|_{L^2(\Omega)} = 0$. A finite set of points $\mathcal{E} = \{x^1, x^2, \cdots, x^N\}$ in $\Omega \subset \mathbb{R}^d$ forms a set of determining nodes if*

$$\lim_{t\to\infty} (u(x^j,t) - v(x^j,t)) = 0, \qquad j = 1, \ldots, N, \tag{3}$$

implies that

$$\lim_{t\to\infty} \|u(t) - v(t)\|_{L^2(\Omega)} = 0.$$

An alternate definition of determining nodes can be given for the H^2-regular case by introducing a standard piecewise linear interpolant $I_h u$ of a function u. To construct the interpolant, a mesh of simplices (triangles in two dimensions or tetrahedra in three) is first built having as vertices the given nodes $\mathcal{E}$, for example by using a Delaunay tessellation. Once the mesh is available, the interpolant can by defined by employing the usual piecewise-linear finite element nodal basis. With such an interpolant, equation (3) in Definition 1 may be replaced with

$$\lim_{t\to\infty} \|I_h u(t) - I_h v(t)\|_{L^2(\Omega)} = 0. \tag{4}$$

One approach to generalizing the notion of determining nodes to the H^1-case is to generalize the interpolant I_h so that it remains valid for H^1-functions; this approach is taken in the present paper, and lower

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bounds for the number N of determining nodes are derived for both two and three dimensions. The bounds take the form

$$N > CG^d,$$

where G is the Grashof number, C is a constant, and d is the spatial dimension. For $d = 2$, the resulting H^1-bound grows quadratically with the Grashof number, whereas the most recent H^2-bound of Jones and Titi [10] grows only linearly with Grashof number. Whether it is possible to show linear growth under only H^1-regularity remains an open question. The generalizations are made possible through the use of some new results in polynomial approximation theory of non-smooth functions in Sobolev spaces.

Outline of the paper. Some background material on the appropriate Sobolev spaces is first presented in §2, and the notation for the remainder of the paper is established. In §3, the weak evolution equation for the determining set function is derived, setting the stage for the analysis in §4 and §6. In §4, some additional required results are summarized, including some inequalities arising from the Sobolev imbedding theorems in special cases, and some extensions to several inequalities originally due to Temam, essential for analyzing and bounding the nonlinear term appearing in weak formulations of the Navier-Stokes equations. In §5, a finite element interpolant due to Scott and Zhang is presented, which will be a key tool in the analysis in §6. This interpolant makes it possible to extend of the notion of determining nodes to the case of H^1-functions with no point-wise values, and to generalize a key interpolation lemma in [10], valid for H^2-functions on the unit square, to the more general setting of H^1-functions on arbitrary polyhedral domains in both two and three dimensions. In §6, a natural generalization of the determining node set is formulated for the H^1-case, and bounds are derived for the number of determining nodes for weak solutions of the Navier-Stokes equations.

2. Spaces and norms.

Some background material on the appropriate Sobolev spaces is presented in this section. The notation for the remainder of the paper is established, following for the most part the notation of [15].

Euclidean d-space is denoted as $\mathbb{R}^d$, a point of which is denoted $\mathbf{x} = (x_1, \ldots, x_d)$, where $x_i \in \mathbb{R}$. The norm in $\mathbb{R}^d$ is denoted as $|\mathbf{x}| = (\sum_{i=1}^d x_i^2)^{1/2}$. The set $\Omega \subset \mathbb{R}^d$ denotes a bounded open subset of $\mathbb{R}^d$, the boundary of which is denoted as Γ. Scalar, vector, and matrix functions over Ω are denoted as $u(\mathbf{x})$; since it will usually be clear when the scalar, vector, or matrix case is intended, a distinction will not be made in the notation. By a *multi-index* α is meant the d-tuple $\alpha = (\alpha_1, \ldots, \alpha_d)$, α_i a nonnegative integer, where $|\alpha| = \sum_{i=1}^d \alpha_i$, which is used to denote mixed partial differentiation of order $|\alpha|$:

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}.$$

Single partial differentiation is denoted as $D_i u = \partial u / \partial x_i$. By defining the vector $\nabla = (D_1, \ldots, D_d)$, and employing the summation convention, the *gradient* and *divergence* operations can be written as tensor products: $(\text{grad } u)_i = (\nabla u)_i = D_i u$, $(\text{grad } u)_{ij} = (\nabla u)_{ij} = D_i u_j$, and $\text{div } u = \nabla \cdot u = D_i u_i$.

The space of k-times continuously differentiable functions defined in Ω is denoted $C^k(\Omega)$, and the subspace of $C^k(\Omega)$ with compact support is denoted $C_0^k(\Omega)$. The Lebesgue spaces are denoted as $L^p(\Omega)$, and are Banach spaces when equipped with the norm $\|u\|_{L^p(\Omega)} = (\int_\Omega |u|^p \, dx)^{1/p}$. Recall that the space $L^2(\Omega)$ is also a Hilbert space when equipped with the inner-product $(u, v) = (u, v)_{L^2(\Omega)} = \int_\Omega uv \, dx$. The Sobolev spaces based on $L^2(\Omega)$ may be defined as

$$H^k(\Omega) = \{ u \in L^2(\Omega) \mid D^\alpha u \in L^2(\Omega), \ 0 \le |\alpha| \le k \},$$

where D^α denotes the weak derivative defined in the usual way. The spaces $H^k(\Omega)$ are Hilbert spaces when equipped with the inner-products and induced norms:

$$(u, v)_{H^k(\Omega)} = \sum_{0 \le |\alpha| \le k} (D^\alpha u, D^\alpha v)_{L^2(\Omega)}, \qquad \|u\|_{H^k(\Omega)} = (u, u)_{H^k(\Omega)}^{1/2}.$$

The norm in $H^k(\Omega)$ can be written in terms of the *semi-norm* $|\cdot|_{H^k(\Omega)}$, in the following way:

$$\|u\|_{H^k(\Omega)}^2 = \sum_{j=0}^k |u|_{H^j(\Omega)}^2, \qquad \text{where} \qquad |u|_{H^k(\Omega)} = \left(\sum_{|\alpha| = k} \|D^\alpha u\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

Note that $H^0(\Omega) = L^2(\Omega)$, and also $(\cdot,\cdot)_{H^0(\Omega)} = (\cdot,\cdot)_{L^2(\Omega)}$ and $|\cdot|_{H^0(\Omega)} = \|\cdot\|_{H^0(\Omega)} = \|\cdot\|_{L^2(\Omega)}$. The following subspace will be important

$$H_0^k(\Omega) = \{u \in H^k(\Omega) \mid D^\alpha u = 0 \ \forall \ \mathbf{x} \in \Gamma, \ 0 \le |\alpha| \le k-1\}.$$

The spaces above extend naturally to vector functions $u = (u_1, u_2, \ldots, u_d)$, which are denoted in boldface (following [15]) as follows:

$$\mathbf{C}_0^\infty(\Omega) = \{C_0^\infty(\Omega)\}^d, \qquad \mathbf{L}^2(\Omega) = \left\{L^2(\Omega)\right\}^d, \qquad \mathbf{H}^k(\Omega) = \left\{H^k(\Omega)\right\}^d, \qquad \mathbf{H}_0^k(\Omega) = \left\{H_0^k(\Omega)\right\}^d.$$

The inner-products and norms in $\mathbf{L}^2(\Omega)$ and (for example) $\mathbf{H}^1(\Omega)$ are extended in the natural way as follows:

$$(u,v) = (u,v)_{L^2(\Omega)} = \sum_{i=1}^d (u_i, v_i), \qquad \|u\|_{L^2(\Omega)} = \left(\sum_{i=1}^d \|u_i\|_{L^2(\Omega)}^2\right)^{1/2}, \qquad |u|_{H^1(\Omega)} = \left(\sum_{i=1}^d |u_i|_{H^1(\Omega)}^2\right)^{1/2},$$

$$\|u\|_{H^1(\Omega)} = \left(\|u\|_{L^2(\Omega)}^2 + |u|_{H^1(\Omega)}^2\right)^{1/2} = \left(\sum_{i=1}^d \|u_i\|_{L^2(\Omega)}^2 + \sum_{i=1}^d |u_i|_{H^1(\Omega)}^2\right)^{1/2}.$$

The Sobolev Imbedding Theorems describe the relationships between the Sobolev spaces and classical functions spaces. To say that a Banach space X is continuously imbedded in a Banach space Y, denoted as $X \hookrightarrow Y$, means that X is a subspace of Y, and that there exists a bounded and linear (hence continuous), one-to-one mapping A from X into Y. If the mapping A is compact (i.e., A maps bounded sets into pre-compact sets), then the imbedding is called *compact*. In §3, some specific imbeddings are considered in more detail. Many of the imbedding results require that the domain Ω be bounded with a locally Lipshitz boundary, denoted as $\Omega \in \mathcal{C}^{0,1}$ (cf. page 67 in [2]). As an example, bounded open convex sets $\Omega \subset \mathbb{R}^d$ satisfy $\Omega \in \mathcal{C}^{0,1}$ (Corollary 1.2.2.3 in [7]). Therefore, convex polyhedral domains, which we restrict our attention to here, are in $\mathcal{C}^{0,1}$.

Define the space of divergence free $\mathbf{C}^\infty$ vector functions $\mathcal{V}$ as

$$\mathcal{V} = \{\phi \in \mathbf{C}_0^\infty(\Omega) \mid \nabla \cdot \phi = 0\},$$

and let

$$H = \text{closure of } \mathcal{V} \text{ in } \mathbf{L}^2(\Omega), \qquad V = \text{closure of } \mathcal{V} \text{ in } \mathbf{H}_0^1(\Omega).$$

The following facts are known about the spaces H and V [15].

THEOREM 2.1. *(Temam) If $\Omega \in \mathcal{C}^{0,1}$, then $V = \left\{u \in \mathbf{H}_0^1(\Omega) \mid \nabla \cdot u = 0\right\}$ and $\mathbf{L}^2(\Omega) = H + H^\perp$, where*

$$H = \left\{u \in \mathbf{L}^2(\Omega) \mid \nabla \cdot u = 0, \quad \text{trace } u = 0\right\}, \qquad H^\perp = \left\{u \in \mathbf{L}^2(\Omega) \mid u = \nabla p, \quad p \in H^1(\Omega)\right\}$$

Proof. See pages 15–18 in [15]. □

3. The evolution equation for the determining function. In this section, the strong and weak formulations of the Navier-Stokes equations are considered, employing the Leray projector, which has the effect of removing the divergence-free constraint from the equation set. The weak evolution equation for the determining set function is then derived, setting the stage for the analysis in §4 and §6.

Define the Leray projector P from the spaces $\mathbf{H}_0^1(\Omega)$ and $\mathbf{L}^2(\Omega)$ onto the spaces V and H, respectively, of divergence free functions, as in [15]. Using the Leray projector, the projected strong form of the Navier-stokes equations (1)–(2) becomes:

$$\frac{du}{dt} + \nu A u + B(u,u) = f, \qquad u(0) = u_0,$$

where $u \in L^2((0,T);V)$, and where the Stokes operator A and bilinear form $B(\cdot,\cdot)$ are defined as

$$Au = -P\Delta u, \qquad B(u,v) = P((u \cdot \nabla)v).$$

To derive the weak formulation, define the bilinear ("Dirichlet") form $a(\cdot,\cdot)$ and trilinear form $b(\cdot,\cdot,\cdot)$ as:

$$a(u,v) = (\nabla u, \nabla v), \qquad b(u,v,w) = (B(u,v),w) = (P((u\cdot\nabla)v),w).$$

Note that $a(u,u) \equiv |u|^2_{H^1(\Omega)}$. Multiplication of the projected strong form equations by a test function $\eta \in V$, and use of a generalized Green's formula in the usual way to produce the bilinear form $a(\cdot,\cdot)$ from the Stokes term νAu, gives rise to the weak formulation.

Definition 2. *Given $f \in L^2((0,T);H)$, if $u \in L^2((0,T);V)$ satisfies*

$$\left(\frac{du}{dt},\eta\right) + \nu a(u,\eta) + b(u,u,\eta) = (f,\eta), \quad \forall \eta \in V, \tag{5}$$

$$u(0) = u_0, \tag{6}$$

then u is called a weak solution of the Navier-Stokes equations.

If u satisfies equation (5), then it can be shown that the initial condition (6) makes sense point-wise (see [15], page 281, for the analysis).

The following symmetries can be shown for the trilinear form.

Lemma 3.1. *It holds that*

1. $b(u,v,v) = 0$
2. $b(u,v,w) = -b(u,w,v)$
3. $b(u-v,u,u-v) = b(u,u,u-v) - b(v,v,u-v)$

Proof. For the proof of properties 1 and 2, see page 163 in [15]. For the proof of property 3, note that

$$b(u,u,u-v) - b(v,v,u-v) = b(u,u,u-v) - b(u-(u-v),u-(u-v),u-v)$$

$$= b(u,u,u-v) - b(u,u,u-v) + b(u-v,u,u-v) + b(u,u-v,u-v) - b(u-v,u-v,u-v)$$

$$= b(u-v,u,u-v) + b(u,u-v,u-v) - b(u-v,u-v,u-v).$$

The second and third terms above must be zero by property 1, so property 3 follows. ∎

For distinct source functions $f,g \in L^2((0,\infty);H)$ and corresponding weak solutions u and v to (5)–(6), consider the difference function $w = u - v$, referred to here as the *determining function.* The weak form equations for w, taking as the test function $\eta \equiv w$, have the form

$$\left(\frac{dw}{dt},w\right) + \nu a(w,w) + b(u,u,w) - b(v,v,w) = (f-g,w).$$

There are several alternate forms for the time derivative term,

$$\left(\frac{dw}{dt},w\right) = \frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(\Omega)} = \|w\|_{L^2(\Omega)}\frac{d}{dt}\|w\|_{L^2(\Omega)}.$$

By Lemma 3.1, $b(w,u,w) = b(u,u,w) - b(v,v,w)$, so the determining function $w = u - v$ must satisfy

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(\Omega)} + \nu|w|^2_{H^1(\Omega)} + b(w,u,w) = (f-g,w), \quad u,v \in V, \quad \Omega \subset \mathbb{R}^d, \quad d = 2,3. \tag{7}$$

This equation, which is an evolution equation for the L^2-norm of the determining function $w = u - v$, will the focus of the analysis in §6.

4. Imbeddings and inequalities. In this section, some additional required results are summarized, including some inequalities arising from the Sobolev imbedding theorems in special cases, and some extensions to several inequalities originally due to Temam, essential for analyzing and bounding the nonlinear term appearing in weak formulations of the Navier-Stokes equations.

Recall first Young's inequality and the Poincare inequality.

Theorem 4.1. *(Young's Inequality) For $a,b \geq 0$, $1 < p,q < \infty$, $1/p + 1/q = 1$, it holds that*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. See page 42 in [11]. □

THEOREM 4.2. *(Poincare Inequality) If Ω is bounded, then it holds that*

$$\|u\|_{L^2(\Omega)} \le \rho(\Omega)|u|_{H^1(\Omega)}, \quad \forall u \in H^1_0(\Omega). \tag{8}$$

Proof. See pages 16–18 in [12]. □

It follows easily that this result extends to the vector case of $u \in \mathbf{H}^1_0(\Omega)$, as do the other classical results for the scalar $L^p(\Omega)$ and Sobolov spaces; see [15] for a detailed exposition. The convention here (as in [15]) will be to subscript the vector norms the same as the scalar case.

The following additional inequalities will be important, which follow from the Sobolev imbedding theorems in special cases.

THEOREM 4.3. *(Sobolev Imbedding Inequalities) Let $\Omega \in \mathcal{C}^{0,1}$. Then for any $u \in \mathbf{H}^1_0(\Omega)$, the following imbedding inequalities hold:*

$$d = 2: \quad \|u\|_{L^q(\Omega)} \le C(q,\Omega)\|u\|_{H^1(\Omega)}, \quad 1 \le q < \infty, \quad \Omega \subset \mathbb{R}^2.$$

$$d = 3: \quad \|u\|_{L^q(\Omega)} \le C(q,\Omega)\|u\|_{H^1(\Omega)}, \quad 1 \le q \le 6, \quad \Omega \subset \mathbb{R}^3.$$

Proof. See page 97 in [2] for a proof, or page 158 in [15] for a discussion. □

The following two less familiar lemmas from [15] will be essential later.

LEMMA 4.4. *(Temam) Let $\Omega \in \mathcal{C}^{0,1}$, and $\Omega \subset \mathbb{R}^2$. Then for any $u \in \mathbf{H}^1_0(\Omega)$, it holds that:*

$$\|u\|_{L^4(\Omega)} \le 2^{1/4}\|u\|^{1/2}_{L^2(\Omega)}|u|^{1/2}_{H^1(\Omega)}.$$

Proof. See page 291 in [15]. □

LEMMA 4.5. *(Temam) Let $\Omega \in \mathcal{C}^{0,1}$, and $\Omega \subset \mathbb{R}^3$. Then for any $u \in \mathbf{H}^1_0(\Omega)$, it holds that:*

$$\|u\|_{L^4(\Omega)} \le 2^{1/2}\|u\|^{1/4}_{L^2(\Omega)}|u|^{3/4}_{H^1(\Omega)}.$$

Proof. See page 296 in [15]. □

A priori bounds can be derived for the nonlinear term $b(\cdot,\cdot,\cdot)$ appearing in (7). If $d = 2$, a classical result is the following bound.

LEMMA 4.6. *(Ladyzhenskaya) Let $\Omega \subset \mathbb{R}^2$. Then the trilinear form $b(u,v,w)$ is bounded on $V \times V \times V$ as follows:*

$$b(u,v,w) \le 2^{1/2}\|u\|^{1/2}_{L^2(\Omega)}|u|^{1/2}_{H^1(\Omega)}|v|_{H^1(\Omega)}\|w\|^{1/2}_{L^2(\Omega)}|w|^{1/2}_{H^1(\Omega)}.$$

Proof. A proof appears on page 292 in [15], employing Lemma 4.4. □

If $d = 3$, the following weaker bound is possible.

LEMMA 4.7. *Let $\Omega \subset \mathbb{R}^3$. Then the trilinear form $b(u,v,w)$ is bounded on $V \times V \times V$ as follows:*

$$b(u,v,w) \le 2\|u\|^{1/4}_{L^2(\Omega)}|u|^{3/4}_{H^1(\Omega)}|v|_{H^1(\Omega)}\|w\|^{1/4}_{L^2(\Omega)}|w|^{3/4}_{H^1(\Omega)}.$$

Proof. By repeated use of Hölder and Schwarz inequalities, it is straight-forward to show that

$$|b(u,v,w)| \le \|u\|_{L^4(\Omega)}|v|_{H^1(\Omega)}\|w\|_{L^4(\Omega)}.$$

Employing the imbedding inequality from Lemma 4.5 twice, we have that

$$|b(u,v,w)| \le 2\|u\|^{1/4}_{L^2(\Omega)}|u|^{3/4}_{H^1(\Omega)}|v|_{H^1(\Omega)}\|w\|^{1/4}_{L^2(\Omega)}|w|^{3/4}_{H^1(\Omega)}.$$

□

In the case that the first and third arguments are the same, we can derive the following alternative bound containing a correction term, which holds in both two and three dimensions.

LEMMA 4.8. *Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$. Then the form $b(w, v, w)$ is bounded on $V \times V \times V$ as*

$$b(w, v, w) \le C_d |v|_{H^1(\Omega)} \left(\|w\|_{L^2(\Omega)} |w|_{H^1(\Omega)} + \lambda_d |w|^2_{H^1(\Omega)} \right),$$

with $\lambda_2 = 0$, $C_2 = 2^{1/2}$, and $\lambda_3 = 1$, $C_3 = C_s^2 \max\{\rho, 1\}$, where C_s is the constant from the Sobolev imbedding inequality, and ρ is the constant from the Poincare inequality.

Proof. The $d = 2$ case is immediate since it is just Lemma 4.6 again, with the first and third argument the same. For the case $d = 3$, we begin as in the proof of Lemma 4.7 with

$$|b(w, v, w)| \le \|w\|_{L^4(\Omega)} |v|_{H^1(\Omega)} \|w\|_{L^4(\Omega)}.$$

We now employ instead the imbedding inequality in Theorem 4.3, which holds in three-dimensions:

$$|b(w, v, w)| \le C_s^2 |v|_{H^1(\Omega)} \|w\|^2_{H^1(\Omega)} = C_s^2 |v|_{H^1(\Omega)} \left(\|w\|^2_{L^2(\Omega)} + |w|^2_{H^1(\Omega)} \right).$$

Employing the Poincare inequality to bound part of the L^2-norm gives

$$|b(w, v, w)| \le C_s^2 |v|_{H^1(\Omega)} \left(\rho \|w\|_{L^2(\Omega)} |w|_{H^1(\Omega)} + |w|^2_{H^1(\Omega)} \right) \le C_3 |v|_{H^1(\Omega)} \left(\|w\|_{L^2(\Omega)} |w|_{H^1(\Omega)} + |w|^2_{H^1(\Omega)} \right),$$

where $C_3 = C_s^2 \max\{\rho, 1\}$. ☐

5. Polynomial interpolation in $\mathbf{H}^1_0$. In this section, a finite element interpolant due to Scott and Zhang is presented, which will be a key tool in the analysis in §6. The interpolant requires only the minimal H^1-regularity for existence and error estimation, and is similar to the usual nodal interpolant and based on the same nodal mesh. This interpolant makes it possible to extend of the notion of determining nodes to the case of H^1-functions with no point-wise values, and to generalize a key interpolation lemma in [10], valid for H^2-functions on the unit square, to the more general setting of H^1-functions on arbitrary polyhedral domains in both two and three dimensions.

Let $\Omega \subset \mathbb{R}^d$ be a d-dimensional polygon, exactly tessellated with quasi-uniform, shape-regular simplices, the vertices of which form N interpolation nodes. Note that for quasi-uniform, shape-regular tessellations in $\mathbb{R}^d$ (see [4] for detailed discussions), it holds that

$$N = C_N h^{-d}, \quad \text{or} \quad h = C_h N^{-1/d},$$

where h is the maximum of the diameters of the simplices, and where C_N and C_h are constants, both of which are independent of both N and h.

In order to discuss interpolation of functions in H^1, recall the Sobolev Imbedding Theorems for the Hilbert scale of spaces H^k, k real and $k \ge 0$. For a more detailed discussion, see [2].

THEOREM 5.1. *(Sobolev Imbedding Theorems) If $\Omega \subset \mathbb{R}^d$ satisfies $\Omega \in \mathcal{C}^{0,1}$, then for nonnegative real numbers k and s it holds that*

$$H^k(\Omega) \hookrightarrow C^s(\bar{\Omega}), \qquad k > s + \frac{d}{2}.$$

For proper definition of a piecewise linear *nodal* interpolant based on the nodes of a tessellation of Ω, $u \in H^1(\Omega)$ must be bounded, so that u exists point-wise. This will be true if u is continuous on Ω, which implies uniform continuity and hence boundedness on $\bar{\Omega}$. For $d = 1$, the Theorem 5.1 shows that $H^1(\Omega)$ is continuously imbedded in $C^0(\bar{\Omega})$, so for one-dimensional problems, the interpolant can be correctly defined. However, in higher dimensions

$$H^{1+\alpha}(\Omega) \hookrightarrow C^0(\bar{\Omega}),$$

only if $\alpha > 0$ when $d = 2$, or if $\alpha > 1/2$ when $d = 3$. This imbedding theorem with $\alpha = 1$ is important in the analysis appearing in [10], and consequently their results require $u \in H^2(\Omega)$.

While it may be possible to use the nodal interpolant and a regularity assumption such as $u \in H^{1+\alpha}(\Omega)$ for $\alpha > 0$ when $d = 2$ and $\alpha > 1/2$ when $d = 3$, an alternative approach is taken here. The interpolant due to Scott and Zhang [14] will be employed, which can be defined correctly for H^1-functions in both two and three spatial dimensions. See [14] for a detailed construction of the interpolant, which is referred to in this paper as the *SZ-interpolant*, and will be denoted as I_h. The SZ-interpolant I_h is constructed from a combination linear interpolation and local averaging on faces and edges of simplices. It can be shown that such a combination produces an interpolant which is a projection from H_0^1 onto V_h, the finite element subspace of $H_0^1(\Omega)$ consisting of continuous piecewise linear polynomials defined over the simplical mesh.

The following interpolation result holds [3, 14, 16] for the SZ-interpolant.

LEMMA 5.2. *(Scott and Zhang) For the SZ-interpolant of $u \in H_0^s(\Omega)$, it holds that*

$$\|u - I_h u\|_{H^m(\Omega)} \le C_i h^{s-m} |u|_{H^s(\Omega)}, \qquad \text{for } m = 0, 1 \quad \text{and } s \ge 1.$$

Proof. See [14]. □

This result extends immediately to vector functions in $\mathbf{H}_0^s$ for $s \ge 1$, and thus to any $u \in V \subset \mathbf{H}_0^1(\Omega)$. In particular,

LEMMA 5.3. *For the SZ-interpolant of $u \in \mathbf{H}_0^{1+\alpha}(\Omega)$, where $\alpha \ge 0$, it holds that*

$$\|u - I_h u\|_{L^2(\Omega)} \le C_i h^{1+\alpha} |u|_{H^{1+\alpha}(\Omega)}.$$

Proof. The proof follows immediately from Lemma 5.2. □

The following lemma is a generalization to $H^{1+\alpha}$, $\alpha \ge 0$, and to general two and three-dimensional polyhedral domains, of the two-dimensional H^2-result in [10], valid for unit square.

LEMMA 5.4. *For $w \in \mathbf{H}_0^{1+\alpha}(\Omega)$, $\alpha \ge 0$, where $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$, it holds that*

$$|w|^2_{H^{1+\alpha}(\Omega)} \ge \frac{N^{2(1+\alpha)/d}}{C_i^2 C_h^2} \|w\|^2_{L^2(\Omega)} - \left[\frac{2N^{(1+\alpha)/d}}{C_i C_h} |w|_{H^{1+\alpha}(\Omega)} \|I_h w\|_{L^2(\Omega)} + \frac{N^{2(1+\alpha)/d}}{C_i^2 C_h^2} \|I_h w\|^2_{L^2(\Omega)} \right]. \tag{9}$$

Proof. Consider

$$\|w\|^2_{L^2(\Omega)} = \|w - I_h w + I_h w\|^2_{L^2(\Omega)} \le (\|w - I_h w\|_{L^2(\Omega)} + \|I_h w\|_{L^2(\Omega)})^2 \le (C_i h^{1+\alpha} |w|_{H^{1+\alpha}(\Omega)} + \|I_h w\|_{L^2(\Omega)})^2$$

$$\le C_i^2 h^{2(1+\alpha)} |w|^2_{H^{1+\alpha}(\Omega)} + 2C_i h^{1+\alpha} |w|_{H^{1+\alpha}(\Omega)} \|I_h w\|_{L^2(\Omega)} + \|I_h w\|^2_{L^2(\Omega)},$$

where Lemma 5.3 has been used. With $h = C_h N^{-1/d}$, it holds that

$$C_i^2 C_h^2 N^{-2(1+\alpha)/d} |w|^2_{H^{1+\alpha}(\Omega)} \ge \|w\|^2_{L^2(\Omega)} - 2C_i C_h N^{-(1+\alpha)/d} |w|_{H^{1+\alpha}(\Omega)} \|I_h w\|_{L^2(\Omega)} - \|I_h w\|^2_{L^2(\Omega)},$$

which gives (9) after division by $C_i^2 C_h^2 N^{-2(1+\alpha)/d}$. □

6. Bounds on the size of the determining set for weak solutions. In this section, a natural generalization of the determining node set is formulated for the H^1-case, and bounds are derived for the number of determining nodes for weak solutions of the Navier-Stokes equations, in both two and three spatial dimensions. The analysis rests essentially on the following tools: Gronwall-type inequalities (which are reviewed); several inequalities for the nonlinear term which were established in §4; two *a priori* L^2- and H^1- bounds on any weak solution (which are provided in two lemmas); suitable use of Young's inequality (following closely the analysis idea of [10]); and the generalized interpolation lemma stated and proved in §5.

One approach to generalizing Definition 1 to the $H_0^1(\Omega)$ case is to generalize the interpolant I_h so that it remains well-defined; we employ the SZ-interpolant I_h of the previous section for this purpose.

DEFINITION 3. *Assume that $u(t)$ is a weak solution of (5)–(6) with source function $f(t)$, and that $v(t)$ is a weak solution of (5)–(6) with source function $g(t)$, where $\lim_{t\to\infty} \|f(t) - g(t)\|_{L^2(\Omega)} = 0$. A finite set of points $\mathcal{E} = \{x^1, x^2, \cdots, x^N\}$ in $\Omega \subset \mathbb{R}^d$ forms a set of determining nodes if*

$$\lim_{t\to\infty} \|I_h u(t) - I_h v(t)\|_{L^2(\Omega)} = 0,$$

implies that

$$\lim_{t\to\infty} \|u(t) - v(t)\|_{L^2(\Omega)} = 0,$$

where $I_h u$ is the SZ-interpolant.

Before getting to the main results, a few more facts must be reviewed. The following inequality, commonly known as Gronwall's inequality [1], is often used in the analysis of differential equations.

LEMMA 6.1. *(Gronwall's Inequality) If $\alpha(t)$ and $\beta(t)$ are real-valued and non-negative on $(0,\infty)$, and if the function $y(t)$ satisfies the following differential inequality:*

$$y'(t)+\alpha(t)y(t)\le\beta(t),\quad \text{a.e. on } (0,\infty),$$

then $y(t)$ is bounded on $(0,\infty)$ by

$$y(t)\le y(0)e^{-\int_0^t\alpha(\tau)d\tau}+\int_0^t\beta(s)e^{-\int_0^t\alpha(\tau)d\tau}ds. \tag{10}$$

Proof. The proof of this classical result seems difficult to locate, so one is given here. Note first that

$$\frac{d}{dt}\left(y(t)e^{\int_0^t\alpha(\tau)d\tau}\right)=y'(t)e^{\int_0^t\alpha(\tau)d\tau}+\alpha(t)y(t)e^{\int_0^t\alpha(\tau)d\tau}\le\beta(t)e^{\int_0^t\alpha(\tau)d\tau}.$$

Thus, it holds that

$$\int_0^t\frac{d}{ds}\left(y(s)e^{\int_0^s\alpha(\tau)d\tau}\right)ds\le\int_0^t\beta(s)e^{\int_0^s\alpha(\tau)d\tau}ds,$$

and so

$$y(t)e^{\int_0^t\alpha(\tau)d\tau}-y(0)\le\int_0^t\beta(s)e^{\int_0^s\alpha(\tau)d\tau}ds,$$

from which (10) follows after a suitable division. □

The following generalized version of the Gronwall inequality will be a key tool in the analysis to follow. The proof may be found in [10]; a similar but weaker generalization is formulated in [6].

LEMMA 6.2. *(Jones and Titi) Let $\alpha(t)$ and $\beta(t)$ be locally integrable and real-valued on $(0,\infty)$, satisfying the following conditions:*

$$\liminf_{t\to\infty}\frac{1}{T}\int_t^{t+T}\alpha(\tau)d\tau=\gamma>0$$

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\alpha^-(\tau)d\tau=\Gamma<\infty$$

$$\lim_{t\to\infty}\frac{1}{T}\int_t^{t+T}\beta^+(\tau)d\tau=0,$$

where $\alpha^-=\max\{-\alpha,0\}$ and $\beta^+=\max\{\beta,0\}$. If $y(t)$ is an absolutely continuous non-negative function on $(0,\infty)$, and $y(t)$ satisfies the following differential inequality:

$$y'(t)+\alpha(t)y(t)\le\beta(t),\quad \text{a.e. on } (0,\infty),$$

then $y(t)\to 0$ as $t\to 0$.

Proof. See [10] for a proof. □

Before getting to the main results, the following two *a priori* bounds on any weak solution must be established.

LEMMA 6.3. *For a weak solution $u\in L^2((0,T);V)$ of the Navier-Stokes equations, with $\Omega\subset\mathbb{R}^d$ and $d=2$ or $d=3$, it holds that*

$$\limsup_{t\to\infty}\|u(t)\|^2_{L^2(\Omega)}\le\frac{\rho^4}{\nu^2}\limsup_{t\to\infty}\|f(t)\|^2_{L^2(\Omega)}, \tag{11}$$

where ρ is the constant from the Poincare inequality.

Proof. Beginning with equation (5) for $\eta = u$, and noting that Lemma 3.1 guarantees that $b(u,u,u) = 0$, it holds that

$$\left(\frac{du}{dt}, u\right) + \nu |u|^2_{H^1(\Omega)} \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}. \tag{12}$$

Since $(\frac{du}{dt}, u) = \frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)}$, it holds that

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + 2\nu |u|^2_{H^1(\Omega)} \leq 2\|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} = \left(\sqrt{\frac{2\rho^2}{\nu}}\|f\|_{L^2(\Omega)}\right)\left(\sqrt{\frac{2\nu}{\rho^2}}\|u\|_{L^2(\Omega)}\right).$$

Employing the Poincare inequality (8) for the H^1-term, and Young's inequality for the right-most term, it holds that

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \frac{2\nu}{\rho^2}\|u\|^2_{L^2(\Omega)} \leq \frac{\rho^2}{\nu}\|f\|^2_{L^2(\Omega)} + \frac{\nu}{\rho^2}\|u\|^2_{L^2(\Omega)},$$

which gives then

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \frac{\nu}{\rho^2}\|u\|^2_{L^2(\Omega)} \leq \frac{\rho^2}{\nu}\|f\|^2_{L^2(\Omega)}.$$

This is a differential inequality for $\|u(t)\|^2_{L^2(\Omega)}$, so that by Gronwall's Inequality (Lemma 6.1) it holds that

$$\|u(t)\|^2_{L^2(\Omega)} \leq \|u(s)\|^2_{L^2(\Omega)} e^{-\int_s^t \nu/\rho^2 d\tau} + \int_s^t \frac{\rho^2}{\nu}\|f(\tau)\|^2 e^{-\int_\tau^t \nu/\rho^2 d\psi} d\tau$$

$$= \|u(s)\|^2_{L^2(\Omega)} e^{-\nu(t-s)/\rho^2} + \int_s^t \frac{\rho^2}{\nu} e^{-\nu(t-\tau)/\rho^2}\|f(\tau)\|^2_{L^2(\Omega)} d\tau$$

$$\leq \|u(s)\|^2_{L^2(\Omega)} e^{-\nu(t-s)/\rho^2} + \frac{\rho^2}{\nu}\sup_{s\leq\delta\leq t}\|f(\delta)\|^2_{L^2(\Omega)} \int_s^t e^{-\nu(t-\tau)/\rho^2} d\tau$$

$$= \|u(s)\|^2_{L^2(\Omega)} e^{-\nu(t-s)/\rho^2} + \frac{\rho^2}{\nu}\sup_{s\leq\delta\leq t}\|f(\delta)\|^2_{L^2(\Omega)} \frac{\rho^2}{\nu}\left(e^0 - e^{-\nu(t-s)/\rho^2}\right),$$

or more simply

$$\|u(t)\|^2_{L^2(\Omega)} \leq \|u(s)\|^2_{L^2(\Omega)} e^{-\nu(t-s)/\rho^2} + \frac{\rho^4}{\nu^2}\sup_{s\leq\delta\leq t}\|f(\delta)\|^2_{L^2(\Omega)},$$

which must hold for every $s \in (0,t]$. Taking the $\limsup_{t\to\infty}$ of both sides of the inequality leaves (11). □

A second estimate is as follows.

LEMMA 6.4. *Let $u \in L^2((0,T);V)$ be a weak solution of the Navier-Stokes equations with $\Omega \subset \mathbb{R}^d$ and $d = 2$ or $d = 3$. Then for every T with $T \geq 1/\nu > 0$ it holds that*

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T} |u(\tau)|^2_{H^1(\Omega)} d\tau \leq \frac{1+2\rho^4}{2\nu^2}\limsup_{t\to\infty}\|f(t)\|^2_{L^2(\Omega)},$$

where ρ is the constant from the Poincare inequality.

Proof. Beginning with equation (12), by Young's inequality it holds that

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + 2\nu |u|^2_{H^1(\Omega)} \leq \left(\sqrt{\frac{2}{\nu}}\|f\|_{L^2(\Omega)}\right)\left(\sqrt{2\nu}\|u\|_{L^2(\Omega)}\right) \leq \frac{1}{\nu}\|f\|^2_{L^2(\Omega)} + \nu\|u\|^2_{L^2(\Omega)}.$$

Integrating from t to $t+T$ with $T>0$ gives

$$\|u(t+T)\|_{L^2(\Omega)}^2 - \|u(t)\|_{L^2(\Omega)}^2 + 2\nu \int_t^{t+T} |u(\tau)|_{H^1(\Omega)}^2 d\tau \le \frac{1}{\nu}\int_t^{t+T} \|f(\tau)\|_{L^2(\Omega)} d\tau + \nu \int_t^{t+T} \|u(\tau)\|_{L^2(\Omega)} d\tau,$$

or rather

$$\int_t^{t+T} |u(\tau)|_{H^1(\Omega)}^2 d\tau \le \frac{1}{2\nu}\|u(t)\|_{L^2(\Omega)}^2 + \frac{T}{2\nu^2}\sup_{t\le s\le T}\|f(s)\|_{L^2(\Omega)}^2 + \frac{T}{2}\sup_{t\le s\le T}\|u(s)\|_{L^2(\Omega)}^2.$$

Taking the $\limsup_{t\to\infty}$ of both sides, and dividing by T, gives

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T} |u(\tau)|_{H^1(\Omega)}^2 d\tau \le \frac{1}{2\nu T}\limsup_{t\to\infty}\|u(t)\|_{L^2(\Omega)}^2 + \frac{1}{2\nu^2}\limsup_{t\to\infty}\|f(t)\|_{L^2(\Omega)}^2 + \frac{1}{2}\limsup_{t\to\infty}\|u(t)\|_{L^2(\Omega)}^2.$$

Using the estimate from Lemma 6.3 twice gives then

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T} |u(\tau)|_{H^1(\Omega)}^2 d\tau \le \left(\frac{\rho^4}{2\nu^3 T} + \frac{1}{2\nu^2} + \frac{\rho^4}{2\nu^2}\right)\limsup_{t\to\infty}\|f(t)\|_{L^2(\Omega)}^2.$$

Since $T \ge 1/\nu > 0$, it holds that

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T} |u|_{H^1(\Omega)}^2 d\tau \le \left(\frac{\rho^4}{2\nu^2} + \frac{1}{2\nu^2} + \frac{\rho^4}{2\nu^2}\right)\limsup_{t\to\infty}\|f(t)\|_{L^2(\Omega)}^2 = \frac{1+2\rho^4}{2\nu^2}\limsup_{t\to\infty}\|f(t)\|_{L^2(\Omega)}^2.$$

$\square$

The main results are now given. This first theorem establishes bounds on the number of determining nodes for weak H^1-solutions of the Navier-Stokes equations in two and three dimensions. The three-dimensional case requires sufficiently large viscosity; this restriction is removed for the three-dimensional case in the second theorem, by doing a more specialized analysis.

THEOREM 6.5. *Let $\Omega \subset \mathbb{R}^d$, $d=2$ or $d=3$, be a polyhedral domain which has been exactly tessellated with a quasi-uniform, shape-regular set of simplices, the vertices of which form a set of N nodes, $\mathcal{E} = \{x^1, x^2, \cdots, x^N\}$. The set $\mathcal{E}$ forms a determining node set for weak H^1-solutions of the Navier-Stokes equations if N is chosen so that*

$$N > C\left(\frac{1}{\nu^2}\limsup_{t\to\infty}\|f(t)\|_{L^2(\Omega)}\right)^d,$$

where C is a constant independent of ν and f. For the case $d=3$, ν must also be sufficiently large for the result to hold.

Proof. Beginning with equation (7), the inequality from Lemma 4.8 is employed along with Cauchy-Schwarz inequality in L^2 to yield the inequality

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2(\Omega)}^2 + \nu|w|_{H^1(\Omega)}^2 \le C_d|u|_{H^1(\Omega)}\left(\|w\|_{L^2(\Omega)}|w|_{H^1(\Omega)} + \lambda_d|w|_{H^1(\Omega)}^2\right) + \|f-g\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)},$$

or equivalently

$$\frac{d}{dt}\|w\|_{L^2(\Omega)}^2 + \left\{(2\nu - 2\lambda_d C_d|u|_{H^1(\Omega)})\left(\frac{|w|_{H^1(\Omega)}}{\|w\|_{L^2(\Omega)}}\right)^2 - 2C_d|u|_{H^1(\Omega)}\left(\frac{|w|_{H^1(\Omega)}}{\|w\|_{L^2(\Omega)}}\right)\right\}\|w\|_{L^2(\Omega)}^2$$

$$\le 2\|f-g\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}.$$

For the second term in braces, by Young's inequality with $p=q=2$ it holds that

$$\left(\frac{1}{\sqrt{2\nu}}2C_d|u|_{H^1(\Omega)}\right)\left(\sqrt{2\nu}\frac{|w|_{H^1(\Omega)}}{\|w\|_{L^2(\Omega)}}\right) \le \frac{2C_d^2}{\nu}|u|_{H^1(\Omega)}^2 + \nu\left(\frac{|w|_{H^1(\Omega)}}{\|w\|_{L^2(\Omega)}}\right)^2.$$

Thus,

$$\frac{d}{dt}\|w\|_{L^2(\Omega)}^2 + \left\{(\nu - 2\lambda_d C_d|u|_{H^1(\Omega)})\frac{|w|_{H^1(\Omega)}^2}{\|w\|_{L^2(\Omega)}^2} - \frac{2C_d^2}{\nu}|u|_{H^1(\Omega)}^2\right\}\|w\|_{L^2(\Omega)}^2 \le 2\|f-g\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}.$$

Assume now that ν is sufficiently large so that the first term in the braces is positive. In particular, assume

$$\nu - 2\lambda_d C_d |u(t)|_{H^1(\Omega)} > \frac{\nu}{2} > 0, \qquad \text{or} \qquad \nu > 4\lambda_d C_d |u(t)|_{H^1(\Omega)}, \qquad t \in (0, \infty).$$

Note that if $d = 2$, then $\lambda_2 = 0$ so that any $\nu > 0$ automatically satisfies the positivity assumption. Therefore, this assumption is only necessary in the three-dimensional case; it will be examined more closely later.

Lemma 5.4 is now employed with the least regularity assumption of $\alpha = 0$ to bound the first term in braces from below (which is possible due to the positivity assumption)

$$\frac{d}{dt}\|w\|^2_{L^2(\Omega)} + \left\{ \frac{\nu}{2\|w\|^2_{L^2(\Omega)}} \left[\frac{N^{2/d}\|w\|^2_{L^2(\Omega)}}{C_i^2 C_h^2} \right] - \frac{2C_d^2}{\nu}|u|^2_{H^1(\Omega)} \right\} \|w\|^2_{L^2(\Omega)} \leq 2\|f-g\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}$$

$$+ \frac{2(\nu - 2\lambda_d C_d |u|_{H^1(\Omega)}) N^{1/d} |w|_{H^1(\Omega)}}{C_i C_h \|w\|^2_{L^2(\Omega)}} \|I_h w\|_{L^2(\Omega)} + \frac{(\nu - 2\lambda_d C_d |u|_{H^1\Omega)}) N^{2/d}}{C_i^2 C_h^2 \|w\|^2_{L^2(\Omega)}} \|I_h w\|^2_{L^2(\Omega)},$$

which is of the form

$$\frac{d}{dt}\|w\|^2_{L^2(\Omega)} + \alpha \|w\|^2_{L^2(\Omega)} \leq \beta,$$

with obvious definition of α and β.

The generalized Gronwall Lemma 6.2 can now be applied. Recall that both $\|f - g\|_{L^2(\Omega)} \to 0$ and $\|I_h w\|_{L^2(\Omega)} \to 0$ as $t \to \infty$ by assumption. Since it is assumed that u and v, and hence w, are in V, so that all other terms appearing in β remain bounded, it must hold that

$$\lim_{t\to\infty} \frac{1}{T}\int_t^{t+T} \beta^+(\tau)d\tau = 0.$$

Since all terms appearing in the expression for α are bounded (by assumption $u \in V$), it holds that

$$\limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} \alpha^-(\tau)d\tau < \infty.$$

It remains to verify that

$$\limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} \alpha(\tau)d\tau > 0. \tag{13}$$

But this is just

$$\frac{\nu N^{2/d}}{2C_i^2 C_h^2} > \limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} \frac{2C_d^2 |u|^2_{H^1(\Omega)}}{\nu} d\tau = \frac{2C_d^2}{\nu} \limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} |u|^2_{H^1(\Omega)} d\tau,$$

or finally

$$N > \left(\frac{4C_d^2 C_i^2 C_h^2 \mathcal{K}}{\nu^2} \right)^{d/2},$$

where by the energy estimate in Proposition 6.4, the parameter $\mathcal{K}$ is bounded by

$$\mathcal{K} = \limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} |u(\tau)|^2_{H^1(\Omega)} d\tau \leq \frac{1+2\rho^4}{2\nu^2} \limsup_{t\to\infty} \|f(t)\|^2_{L^2(\Omega)} \leq \frac{1+2\rho^4}{2} \left(\limsup_{t\to\infty} \frac{\|f(t)\|_{L^2(\Omega)}}{\nu} \right)^2,$$

recalling (cf. [13], page 12) that if $a(t), b(t) \geq 0$, $\limsup_{t\to\infty} a(t)b(t) \leq (\limsup_{t\to\infty} a(t))\,(\limsup_{t\to\infty} b(t))$. Therefore, if

$$N > \left(2C_d^2 C_i^2 C_h^2 (1+2\rho^4)\right)^{d/2} \left(\limsup_{t\to\infty} \frac{\|f(t)\|_{L^2(\Omega)}}{\nu^2} \right)^d, \tag{14}$$

implying that (13) holds, then by the Gronwall Lemma 6.2, it follows that

$$\lim_{t\to\infty} \|w(t)\|_{L^2(\Omega)} = \lim_{t\to\infty} \|u(t) - v(t)\|_{L^2(\Omega)} = 0.$$

□

The restriction of sufficiently large viscosity is now removed for the three-dimensional case, by employing a different analysis approach; however, the penalty is a more rapid growth in N as a function of the inverse of the viscosity ν. In order to prove the result, the following *a priori* bound is required, for which we have not developed a proof at this time.

PROPOSITION 6.6. *Let $u \in L^2((0,T);V)$ be a weak solution of the Navier-Stokes equations with $\Omega \subset \mathbb{R}^d$ and $d = 2$ or $d = 3$. Then for every T with $T \geq 1/\nu > 0$ it holds that*

$$\limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} |u(\tau)|^4_{H^1(\Omega)} d\tau \leq \frac{C_a}{\nu^4} \limsup_{t\to\infty} \|f(t)\|^4_{L^2(\Omega)}.$$

If this *a priori* bound holds, then we have the following result for $d = 3$.

THEOREM 6.7. *Let $\Omega \subset \mathbb{R}^3$ be a polyhedral domain which has been exactly tessellated with a quasi-uniform, shape-regular set of simplices, the vertices of which form a set of N nodes, $\mathcal{E} = \{x^1, x^2, \cdots, x^N\}$. The set $\mathcal{E}$ forms a determining node set for weak H^1-solutions of the Navier-Stokes equations if N is chosen so that*

$$N > C\left(\frac{1}{\nu^2}\limsup_{t\to\infty} \|f(t)\|_{L^2(\Omega)}\right)^6,$$

where C is constant independent of ν and f.

Proof. Beginning as in the proof of Theorem 6.5, but employing the inequality from Lemma 4.7 rather than the inequality from Lemma 4.8, gives

$$\frac{d}{dt}\|w\|^2_{L^2(\Omega)} + 2\nu|w|^2_{H^1(\Omega)} \leq 4|u|_{H^1(\Omega)}\|w\|^{1/2}_{L^2(\Omega)}|w|^{3/2}_{H^1(\Omega)} + 2\|f-g\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)},$$

or equivalently

$$\frac{d}{dt}\|w\|^2_{L^2(\Omega)} + \left\{2\nu\left(\frac{|w|_{H^1(\Omega)}}{\|w\|_{L^2(\Omega)}}\right)^2 - 4|u|_{H^1(\Omega)}\left(\frac{|w|_{H^1(\Omega)}}{\|w\|_{L^2(\Omega)}}\right)^{3/2}\right\}\|w\|^2_{L^2(\Omega)} \leq 2\|f-g\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}.$$

For the second term in braces, again employing Young's inequality, but with $p = 4$ and $q = 4/3$, gives

$$4|u|_{H^1(\Omega)}\left(\frac{|w|_{H^1(\Omega)}}{\|w\|_{L^2(\Omega)}}\right)^{3/2} = \left(4\left(\frac{3}{4\nu}\right)^{3/4}|u|_{H^1(\Omega)}\right)\left(\left(\frac{4\nu}{3}\right)^{3/4}\left(\frac{|w|_{H^1(\Omega)}}{\|w\|_{L^2(\Omega)}}\right)^{3/2}\right)$$

$$\leq \frac{\left(4\left(\frac{3}{4\nu}\right)^{3/4}|u|_{H^1(\Omega)}\right)^4}{4} + \frac{\left(\left(\frac{4\nu}{3}\right)^{3/4}\left(\frac{|w|_{H^1(\Omega)}}{\|w\|_{L^2(\Omega)}}\right)^{3/2}\right)^{4/3}}{4/3} = \frac{3^3}{\nu^3}|u|^4_{H^1(\Omega)} + \nu\left(\frac{|w|_{H^1(\Omega)}}{\|w\|_{L^2(\Omega)}}\right)^2.$$

Using this to bound the term in braces below so that

$$\frac{d}{dt}\|w\|^2_{L^2(\Omega)} + \left\{\nu\left(\frac{|w|_{H^1(\Omega)}}{\|w\|_{L^2(\Omega)}}\right)^2 - \frac{3^3}{\nu^3}|u|^4_{H^1(\Omega)}\right\}\|w\|^2_{L^2(\Omega)} \leq 2\|f-g\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}.$$

The analysis now proceeds exactly as in Theorem 6.5, except for verification of the condition:

$$\limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} \alpha(\tau)d\tau > 0. \tag{15}$$

But this is now

$$\frac{\nu N^{2/d}}{2C_i^2 C_h^2} > \limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\frac{3^3|u|^4_{H^1(\Omega)}}{\nu^3}d\tau = \frac{3^3}{\nu^3}\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}|u|^4_{H^1(\Omega)}d\tau,$$

where $d = 3$, or finally

$$N > \left(\frac{3^3 2 C_i^2 C_h^2 \mathcal{K}}{\nu^4}\right)^{3/2},$$

where by the energy estimate in Lemma 6.6, the parameter $\mathcal{K}$ is bounded by

$$\mathcal{K} = \limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} |u(\tau)|_{H^1(\Omega)}^4 d\tau \le \frac{C_a}{\nu^4} \limsup_{t\to\infty} \|f(t)\|_{L^2(\Omega)}^4 \le C_a \left(\limsup_{t\to\infty} \frac{\|f(t)\|_{L^2(\Omega)}}{\nu}\right)^4.$$

Therefore, if

$$N > \left(3^3 2 C_i^2 C_h^2 C_a\right)^{3/2} \left(\limsup_{t\to\infty} \frac{\|f(t)\|_{L^2(\Omega)}}{\nu}\right)^6, \tag{16}$$

implying that (15) holds, then by the Gronwall Lemma 6.2, it follows that

$$\lim_{t\to\infty} \|w(t)\|_{L^2(\Omega)} = \lim_{t\to\infty} \|u(t) - v(t)\|_{L^2(\Omega)} = 0.$$

□

Now, let F be defined as

$$F = \limsup_{t\to\infty} \left(\int_\Omega |f(x,t)|^2\right)^{1/2}.$$

Note that if $f(x,t) = f(x)$, then $F = \|f\|_{L^2(\Omega)}$.

Definition 4. *The generalized Grashof number G is defined as*

$$G = \frac{F}{\nu^2}.$$

Note that this definition differs somewhat from that of [10]. The generalized Grashof number Gr defined in [10], motivated by analysis of strong solutions, is related to G above as

$$Gr = \frac{F}{\lambda_1 \nu^2} = \frac{1}{\lambda_1} G,$$

where λ_1 is the smallest eigenvalue of the Stokes operator. Note that by working with the weak form, the need for λ_1 is avoided in the analysis (although ρ from the Poincare inequality now plays a similar role), and hence G is defined to be independent of λ_1. This definition of G has been used in some existence and uniqueness theories for the Navier-Stokes equations (e.g., see page 331 in [4]).

Corollary 6.8. *Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$, be a polyhedral domain which has been exactly tessellated with a quasi-uniform, shape-regular set of simplices, the vertices of which form a set of N nodes, $\mathcal{E} = \{x^1, x^2, \cdots, x^N\}$. The set $\mathcal{E}$ forms a determining node set for weak H^1-solutions of the Navier-Stokes equations if N is chosen so that*

$$N > CG^d,$$

where C is constant independent of ν and f, and where G is the generalized Grashof number above. For the case $d = 3$, it must also hold that the viscosity is sufficiently large. An alternative bound for the case $d = 3$ with no restriction on the viscosity is

$$N > CG^6.$$

Proof. This follows immediately from Theorems 6.5 and 6.7. □

7. Concluding remarks. New upper bounds on the dimension of the determining set were derived for weak solutions of the Navier-Stokes equations. These results extend the recent bounds due to Jones and Titi in three ways. First, the bounds were derived under the minimal H^1-regularity required to define a weak solution of the Navier-Stokes equations. Second, the new bounds are valid for arbitrary polyhedral domains, whereas previous results were derived for the unit square. Third, the results hold also in the three-dimensional case, whereas previous bounds were restricted to two dimensions.

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