AN ODYSSEY INTO LOCAL REFINEMENT AND MULTILEVEL PRECONDITIONING I: OPTIMALITY OF THE BPX PRECONDITIONER *

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Abstract. In this article, we examine the Bramble-Pasciak-Xu (BPX) preconditioner in the setting of local 3D mesh refinement. While the available optimality results for the BPX preconditioner have been constructed primarily in the setting of uniformly refined meshes, a notable exception is the 2D result due to Dahmen and Kunoth, which established BPX optimality on meshes produced by a local 2D red-green refinement. The purpose of this article is to extend this original 2D optimality result to the local 3D red-green refinement procedure introduced by Bornemann-Erdmann-Kornhuber (BEK). The extension is reduced to establishing that locally enriched finite element subspaces allow for the construction of a scaled basis which is formally Riesz stable. This construction turns out to rest not only on shape regularity of the refined elements, but also critically on a number of geometrical properties we establish between neighboring simplices produced by the BEK refinement procedure. We also show that the number of degrees of freedom used for smoothing is bounded by a constant times the number of degrees of freedom introduced at that level of refinement, indicating that a practical implementable version of the resulting BPX preconditioner for the BEK refinement setting has provably optimal (linear) computational complexity per iteration, as well as having a uniformly bounded condition number. The theoretical framework supports arbitrary spatial dimension $d \geq 1$, and we indicate clearly which geometrical properties established here must be re-established to show BPX optimality for spatial dimension $d \geq 4$. The proof techniques require no coefficient smoothness assumptions beyond those required for well-posedness in H^1.

Key words. finite element approximation theory, multilevel preconditioning, BPX, three dimensions, local mesh refinement, red-green refinement.

AMS subject classifications. 65M55, 65N55, 65N22, 65F10

1. Introduction. In this article, we analyze the impact of local mesh refinement on the stability of multilevel finite element spaces and on the optimality (linear space and time complexity) of multilevel preconditioners. Adaptive refinement techniques have become a crucial tool for many applications, and access to optimal or near-optimal multilevel preconditioners for locally refined mesh situations is of primary concern to computational scientists. The preconditioners which can be expected to have somewhat favorable space and time complexity in such local refinement scenarios are the hierarchical basis (HB) method, the Bramble-Pasciak-Xu (BPX) preconditioner, and the wavelet modified (or stabilized) hierarchical basis (WHB) method. While there are optimality results for both the BPX and WHB preconditioners in the literature, these are primarily for quasiuniform meshes and/or two space dimensions (with some exceptions noted below). In particular, there are few hard results in the literature on the optimality of these methods for various realistic local mesh refinement hierarchies, especially in three space dimensions. In this article, the first in a series of three articles [2, 3] on local refinement and multilevel preconditioners, we

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assemble optimality results for the BPX preconditioner in local refinement scenarios in three spacial dimensions. (The material forming this trilogy is based on the first author's Ph.D. dissertation [1].) The second article [3] builds on the BPX results we present here to develop some analogous optimality results for the WHB method in local refinement settings. The main results in both articles are valid for any spatial dimension $d \geq 1$, for nonsmooth PDE coefficients $p \in L_\infty(\Omega)$.

The problem class we focus on here and in [3] is linear second order partial differential equations (PDE) of the form:

$$-\nabla \cdot (p \ \nabla u) + q \ u = f, \quad u \in H_0^1(\Omega). \tag{1.1}$$

Here, $f \in L_2(\Omega)$, $p, q \in L_\infty(\Omega)$, $p : \Omega \to L(\mathbb{R}^d, \mathbb{R}^d)$, $q : \Omega \to \mathbb{R}$, where p is a symmetric positive definite matrix function, and where q is a nonnegative function. Let $\mathcal{T}_0$ be a shape regular and quasiuniform initial partition of Ω into a finite number of d simplices, and generate $\mathcal{T}_1, \mathcal{T}_2, \ldots$ by refining the initial partition using red-green local refinement strategies in $d = 3$ spatial dimensions. Denote as $\mathcal{S}_j$ the simplicial linear C^0 finite element space corresponding to $\mathcal{T}_j$ equipped with zero boundary values. The set of nodal basis functions for $\mathcal{S}_j$ is denoted by $\Phi^{(j)} = \{\phi_i^{(j)}\}_{i=1}^{N_j}$ where $N_j = \dim \mathcal{S}_j$ is equal to the number of interior nodes in $\mathcal{T}_j$, representing the number of degrees of freedom in the discrete space. Successively refined finite element spaces will form the following nested sequence:

$$\mathcal{S}_0 \subset \mathcal{S}_1 \subset \ldots \subset \mathcal{S}_j \subset \ldots \subset H_0^1(\Omega). \tag{1.2}$$

Let the bilinear form and the functional associated with the weak formulation of (1.1) be denoted as

$$a(u, v) = \int_\Omega p \ \nabla u \cdot \nabla v + q \ u \ v \ dx, \quad b(v) = \int_\Omega f \ v \ dx, \quad u, v \in H_0^1(\Omega).$$

We consider primarily the following Galerkin formulation: Find $u \in \mathcal{S}_j$, such that

$$a(u, v) = b(v), \quad \forall v \in \mathcal{S}_j. \tag{1.3}$$

The finite element approximation in $\mathcal{S}_j$ has the form $u^{(j)} = \sum_{i=1}^{N_j} u_i \phi_i^{(j)}$, where $u = (u_1, \ldots, u_{N_j})^T$ denotes the coefficients of $u^{(j)}$ with respect to $\Phi^{(j)}$. The resulting *discretization operator* $A^{(j)} = \{a(\phi_k^{(j)}, \phi_l^{(j)})\}_{k,l=1}^{N_j}$ must be inverted numerically to determine the coefficients u from the linear system:

$$A^{(j)} u = F^{(j)}, \tag{1.4}$$

where $F^{(j)} = \{b(\phi_l^{(j)})\}_{l=1}^{N_j}$. Our task is to solve (1.4) with optimal (linear) complexity in both storage and computation, where the finite element spaces $\mathcal{S}_j$ are built on locally refined meshes.

Optimality of the BPX preconditioner with generic local refinement was shown by Bramble and Pasciak [8], where the impact of the local smoother and the local projection operator on the estimates was carefully analyzed. The two primary results on optimality of BPX preconditioner in the local refinement settings are due to Dahmen and Kunoth [11] and Bornemann and Yserentant [6]. Both works consider only two space dimensions, and in particular, the refinement strategies analyzed are restricted 2D red-green refinement and 2D red refinement, respectively. In this paper, we extend the framework developed in [11] to a practical, implementable 3D local red-green refinement procedure introduced by Bornemann-Erdmann-Kornhuber (BEK) [5]. We will refer to this as the BEK refinement procedure.

There is one main and one side result of this article. The main one establishes that the BPX preconditioner is optimal–both norm equivalence to H^1-norm and computational complexity per iteration–for the resulting locally refined 3D finite element hierarchy. The analysis here heavily relies on the techniques of the Dahmen-Kunoth [11] framework and can be seen as an extension to three spatial dimensions with the realistic BEK refinement procedure [5] being the application of interest.

The side result is the H^1-stability of L_2-projection onto finite element spaces built through the BEK local refinement procedure. This question is currently under intensive study in the finite element community due to its relationship to multilevel preconditioning. The existing theoretical results, due primarily to Carstensen [10] and Bramble-Pasciak-Steinbach [9] involve *a posteriori* verification of somewhat complicated mesh conditions after local refinement has taken place. If such mesh conditions are not satisfied, one has to redefine the mesh. However, an interesting consequence of the BPX optimality results for locally refined 2D and 3D meshes established here is H^1-stability of L_2-projection restricted to the same locally enriched finite element spaces. This result, which is established in [3] based on the results here, appears to be the first *a priori* H^1-stability result for L_2-projection on finite element spaces produced by practical and easily implementable 2D and 3D local refinement procedures.

Outline of the paper. In §2, we outline the BPX preconditioner in the local refinement setting. In §3, we introduce the BPX and slice operator to make the connection between the (implementable) BPX preconditioner and the corresponding slice operator which is used in the norm equivalence to H^1-norm. In §4, we introduce some basic approximation theory tools used in the analysis such as Besov spaces and *Bernstein estimates*. The framework for the main norm equivalence is also established here. In §5, we list the BEK refinement conditions. We give several theorems about the generation and size relations of the neighboring simplices, thereby establishing quasiuniformity for the simplices in the support of a basis function. In §6, we explicitly give an upper bound for the nodes introduced in the refinement region. This implies that one application of the BPX preconditioner to a function has linear (optimal) computational complexity. In §7, we use the geometrical results from §5 to extend the 2D Dahmen-Kunoth results to the 3D BEK refinement procedure by establishing the desired norm equivalence. The geometrical properties established in §5 lead to quasiuniformity of the support which gives rise to an L_2-stable Riesz basis; one can then establish the Bernstein estimate. While it is not possible to establish a Jackson estimate due to the nature of local adaptivity, in §8 the remaining inequality in the norm equivalence is handled directly using approximation theory tools, as in the original work [11]. We conclude in §9.

2. Preliminaries. In the uniform refinement setting, the parallelized or additive version of the multigrid method, also known as the BPX preconditioner, is defined as follows:

$$Xu := \sum_{j=0}^{J} 2^{j(d-2)} \sum_{i=1}^{N_j} (u, \phi_i^{(j)}) \phi_i^{(j)}.$$

In the local refinement setting, in order to maintain optimal computational complexity, the smoother is restricted to a local space $\tilde{\mathcal{S}}_j$, where typically

$$\mathcal{S}_j \setminus \mathcal{S}_{j-1} \subset \tilde{\mathcal{S}}_j \subset \mathcal{S}_j.$$

The basic restriction on the refinement procedure is that it remains *nested*. In other words, tetrahedra of level j which are not candidates for further refinement will never be touched in the future. Let Ω_j denote the refinement region, namely, the union of the supports of basis functions which are introduced at level j. Due to nested refinement $\Omega_j \subset \Omega_{j-1}$. Then the following hierarchy holds:

$$\Omega_J \subset \Omega_{J-1} \subset \cdots \subset \Omega_0 = \Omega. \tag{2.1}$$

The main ingredient in the local refinement setting for the analysis of the BPX preconditioner is the *local quasi-interpolant* which will be explicitly given in (8.5):

$$\tilde{Q}_j : L_2(\Omega) \to \mathcal{S}_j. \tag{2.2}$$

In fact, in our construction this operator will turn out to be a *projection*, i.e. L_2-self-adjoint and $\tilde{Q}_j^2 = \tilde{Q}_j$. For $u \in \mathcal{S}_J$, the projection $\tilde{Q}_j$ will have the *local* property that $(\tilde{Q}_j - \tilde{Q}_{j-1})u$ vanishes outside of Ω_j. A detailed discussion on the vanishing property is also presented by Oswald (see page 94 in [19]). Let the *local smoother* be the following symmetric positive definite operator (examples of these are given in [7]):

$$\tilde{R}_j : \tilde{\mathcal{S}}_j \to \tilde{\mathcal{S}}_j, \tag{2.3}$$

where $\tilde{\mathcal{S}}_j := (\tilde{Q}_j - \tilde{Q}_{j-1})\mathcal{S}_J$. This choice for $\tilde{\mathcal{S}}_j$ indicates that the smoother acts on a local collection of degrees of freedom (DOF) which will give rise to optimal computational complexity per iteration. Nodes–equivalently, DOF–corresponding to $\tilde{\mathcal{S}}_j$ and their cardinality will be denoted $\tilde{\mathcal{N}}_j$ and $\tilde{N}_j$, respectively. We now define the BPX preconditioner for the local refinement setting as:

$$Xu := \sum_{j=0}^{J} 2^{j(d-2)} \sum_{i \in \tilde{\mathcal{N}}_j} (u, \phi_i^{(j)})\phi_i^{(j)}. \tag{2.4}$$

The multilevel splitting of $u \in \mathcal{S}_J$ using (2.2) is then

$$u = \sum_{j=0}^{J} (\tilde{Q}_j - \tilde{Q}_{j-1})u, \tag{2.5}$$

with $\tilde{Q}_{-1} = 0$ and $\tilde{Q}_J$ the identity on $\mathcal{S}_J$. If $\tilde{Q}_j$ is a local projection such as a local quasi-interpolant, then the individual terms in this splitting are locally supported. The main difference in the analysis between the local and uniform refinement cases lies in the choice of the projection. Namely, in the uniform refinement case, the L_2-projection Q_j is used for the splitting in (2.5). But since $(Q_j - Q_{j-1})u$ has global support, it is not a practical choice that will lead to an optimal method. Therefore, in the local refinement case, we employ the *local projection* $\tilde{Q}_j$ which allows for optimal computational complexity. While the analysis throughout the paper will be based on the use of $\tilde{Q}_j$, with the exception of the optimal computational complexity result in (6.1), all results also hold for projections which are globally supported such as Q_j. In particular, the main optimal norm equivalence results (4.1) and (8.12) hold for Q_j as well as $\tilde{Q}_j$. This fact will be used in [3] for the analysis of H^1-stability of Q_j.

3. The BPX and slice operators. Throughout this article we use the following standard notation: for $x, y \in \mathbb{R}^+$ and universal constants $c_1, c_2 \in \mathbb{R}^+$, we write:

$$x \eqsim y \quad \text{if} \quad c_1 y \leq x \leq c_2 y.$$

We first move to a general analysis framework where (2.4) becomes a special case. From this point on, the following operator will be referred as the BPX preconditioner for $u \in \mathcal{S}_J$.

$$Bu := \sum_{j=0}^{J} \tilde{R}_j(\tilde{Q}_j - \tilde{Q}_{j-1})u, \tag{3.1}$$

Utilizing the splitting (2.5), one can write $u = \sum_{j=0}^{J} u^{(j)^f}$, where $u^{(j)^f} := (\tilde{Q}_j - \tilde{Q}_{j-1})u$. Note that B can be written as a diagonal operator using (2.5):

$$B = \operatorname{diag}(\tilde{R}_0\tilde{Q}_0, \tilde{R}_1(\tilde{Q}_1 - \tilde{Q}_0), \ldots, \tilde{R}_J(\tilde{Q}_J - \tilde{Q}_{J-1})).$$

Using the projection properties, one can observe that

$$\tilde{R}_j(\tilde{Q}_j - \tilde{Q}_{j-1})\tilde{R}_j^{-1}(\tilde{Q}_j - \tilde{Q}_{j-1})u^{(j)^f} = u^{(j)^f}.$$

Then,

$$B^{-1} = \operatorname{diag}(\tilde{R}_0^{-1}\tilde{Q}_0, \tilde{R}_1^{-1}(\tilde{Q}_1 - \tilde{Q}_0), \ldots, \tilde{R}_J^{-1}(\tilde{Q}_J - \tilde{Q}_{J-1})).$$

The ultimate goal is the prove the following norm equivalence

$$(B^{-1}u, u) \eqsim (Au, u), \tag{3.2}$$

which will give $\kappa(BA) \eqsim 1$. To reach this goal, we use the *slice operator* induced by the splitting (2.5):

$$Cu := \sum_{j=0}^{J} 2^{2j}(\tilde{Q}_j - \tilde{Q}_{j-1})u. \tag{3.3}$$

The following assumption (which naturally holds) will be enforced on the local smoothing operator $\tilde{R}_j$:

$$2^{-2j}\|v\|^2 \eqsim (\tilde{R}_j v, v), \quad v \in \tilde{\mathcal{S}}_j. \tag{3.4}$$

Now, we can link the two operators by using (3.4).

$$\begin{aligned}(B^{-1}u, u) &= \sum_{j=0}^{J}(\tilde{R}_j^{-1}(\tilde{Q}_j - \tilde{Q}_{j-1})u, (\tilde{Q}_j - \tilde{Q}_{j-1})u) \\ &\eqsim \sum_{j=0}^{J} 2^{2j}\|(\tilde{Q}_j - \tilde{Q}_{j-1})u\|_{L_2}^2 = (Cu, u).\end{aligned} \tag{3.5}$$

Therefore, we focus entirely on establishing the following norm equivalence:

$$(Cu, u) \eqsim (Au, u). \tag{3.6}$$

The flow of this article is as follows. The proof of the main norm equivalence (3.6) will be given in Theorem 4.1.

$$\frac{c_1}{\tilde{v}_J^2}(Cu, u) \leq (Au, u) \leq c_2(Cu, u), \quad u \in \mathcal{S}_J.$$

The missing crucial result of this article, i.e. $\tilde{v}_J = O(1)$ as $J \to \infty$, will be shown in (8.12) so that (3.6) holds.

4. Main norm equivalence. Let $\Omega \subset \mathbb{R}^d$ be open, for arbitrary $k = 1, 2, \ldots$ and $h \in \mathbb{R}^d$, and define the subset

$$\Omega_{k,h} = \left\{ x \in \mathbb{R}^d : [x, x + kh] \subset \Omega \right\}.$$

Define the directional k-th order difference of $f \in L_p(\Omega)$ as

$$\left(\Delta_h^k f\right)(x) := \sum_{r=0}^{k} \left(\begin{array}{c} k \\ r \end{array} \right) (-1)^{k-r} f(x + rh), \quad x, h \in \mathbb{R}^d.$$

A finer scale of smoothness than differentiability, *modulus of smoothness*, is a central tool in the analysis. This will be used in the definition of Besov spaces. L_p-modulus of smoothness is defined as

$$\omega_k(f, t, \Omega)_p := \sup_{|h| \le t} \|\Delta_h^k f\|_{L_p(\Omega_{k,h})}.$$

Then, Besov spaces are defined to be the collection of functions $f \in L_p(\Omega)$ with a finite Besov norm defined as follows:

$$\|f\|^q_{B^s_{p,q}(\Omega)} := \|f\|^q_{L_p(\Omega)} + |f|^q_{B^s_{p,q}(\Omega)},$$

where the seminorm is given by

$$|f|_{B^s_{p,q}(\Omega)} := \|\{2^{sj}\omega_k(f, 2^{-j}, \Omega)_p\}_{j\in\mathbb{N}_0}\|_{l_q},$$

and k is any fixed integer larger than s.

Besov space becomes the primary function space setting by realizing the Sobolev space as a Besov space:

$$H^s(\Omega) \cong B^s_{2,2}(\Omega), \quad s > 0.$$

The analysis needed for functions in Sobolev space is done in the Besov space. The primary motivation for employing the Besov space stems from the fact that the characterization of functions which have a given upper bound for the error of approximation sometimes calls for a finer scale of smoothness.

The Bernstein estimate is defined as:

$$\omega_{k+1}(u, t)_p \le c\ (\min\{1, t2^J\})^\beta \|u\|_{L_p}, \quad u \in \mathcal{S}_J, \tag{4.1}$$

where c is independent of u and J. Usually $k =$ degree of the element and in the case of linear finite elements $k = 1$. Here β is determined by the global smoothness of the approximation space as well as p. For C^r finite elements, $\beta = \min\{1 + r + \frac{1}{p}, k + 1\}$.

Let $\tilde{v}_J$ be defined as follows.

$$\tilde{v}_{j,J} := \sup_{u\in\mathcal{S}_J} \frac{\|u - \tilde{Q}_j u\|_{L_2}}{\omega_2(u, 2^{-j})_2}, \quad \tilde{v}_J := \max\left\{1, \tilde{v}_{j,J} : j = 0, \ldots, J\right\}. \tag{4.2}$$

Following [11] we have then

THEOREM 4.1. *Suppose the Bernstein estimate (4.1) holds for some real number $\beta > 1$. Then, for each $0 < s < \min\{\beta, 2\}$, there exist constants $0 < c_1,\ c_2 < \infty$ independent of $u \in \mathcal{S}_J,\ J = 0, 1, \ldots,$ such that the following norm equivalence holds:*

$$\frac{c_1}{\tilde{v}_J^2}(Cu, u) \le \|u\|^2_{B^1_{2,2}} \le c_2(Cu, u), \quad u \in \mathcal{S}_J. \tag{4.3}$$

Proof. Adding and subtracting u gives

$$(Cu, u) \le c\ (\|u\|_{L_2}^2 + \sum_{j=0}^{J} 2^{2j} \|\tilde{Q}_j u - u\|_{L_2}^2).$$

Using the definition of $\tilde{v}_J$ we have

$$\|\tilde{Q}_j u - u\|_{L_2}^2 \le c\ \tilde{v}_J\ \omega_2(u, 2^{-j})_2,$$

and thus

$$(Cu, u) \le c v_J^2\ (\|u\|_{L_2}^2 + \sum_{j=0}^{J} 2^{2j} \omega_2(u, 2^{-j})_2^2) \le c v_J^2 \|u\|_{B_{2,2}^1}^2.$$

For the upper bound, we use the splitting (2.5) where $u_j := (\tilde{Q}_j - \tilde{Q}_{j-1})u$. By (4.1),

$$\omega_2(u, 2^{-n})_2 \le c\ \sum_{j=0}^{\infty} \omega_2(u_j, 2^{-n})_2 \le c\ (\sum_{j=0}^{n} 2^{(j-n)\gamma} \|u_j\|_{L_2} + \sum_{j=n+1}^{\infty} \|u_j\|_{L_2}),$$

which gives

$$\begin{aligned}\sum_{n=0}^{\infty} 2^{2n} \omega_2(u, 2^{-n})_2^2 &\le c\ (\sum_{n=0}^{\infty} 2^{2n} \{2^{-n\gamma} \sum_{j=0}^{n} 2^{j\gamma} \|u_j\|_{L_2} + \sum_{j=n+1}^{\infty} \|u_j\|_{L_2}\}^2) \\ &\le c\ (\sum_{n=0}^{\infty} 2^{2n(1-\gamma)} \{\sum_{j=0}^{n} 2^{j\gamma} \|u_j\|_{L_2}\}^2 + \sum_{n=0}^{\infty} 2^{2n} \{\sum_{j=n+1}^{\infty} \|u_j\|_{L_2}\}^2) \\ &\le c\ (\sum_{n=0}^{\infty} 2^{2n} \|u_n\|_{L_2}^2),\end{aligned}$$

with the last step due to a Hardy inequality (cf. [11]). This gives $\|u\|_{B_{2,2}^1}^2 \le c\ (Cu, u)$. ∎

5. The BEK refinement procedure. Our interest is to show optimality of the BPX preconditioner for the local 3D red-green refinement introduced by Bornemann-Erdmann-Kornhuber [5]. This 3D red-green refinement is practical, easy to implement, and numerical experiments were presented in [5]. A similar refinement procedure was analyzed by Bey [4]; in particular, the same green closure strategy was used in both papers. While these refinement procedures are known to be asymptocially non-degenerate (and thus produce shape regular simplices at every level of refinement), shape regularity is insufficient to construct a stable Riesz basis for finite element spaces on locally adapted meshes. To construct a stable Riesz basis we will need to establish patchwise quasiuniformity as in [11]; as a result, d-vertex adjacency relationships that are independent of shape regularity of the elements must be established between neighboring tetrahedra as done in [11] for triangles.

We first list a number of geometric assumptions we make concerning the underlying mesh. Let $\Omega \subset \mathbb{R}^3$ be a polyhedral domain. We assume that the triangulation $\mathcal{T}_j$ of Ω at level j is a collection of tetrahedra with mutually disjoint interiors which cover $\Omega = \bigcup_{\tau \in \mathcal{T}_j} \tau$. We want to generate successive refinements $\mathcal{T}_0, \mathcal{T}_1, \ldots$ which satisfy the following conditions:

Assumption 5.1. **Nestedness:** *Each tetrahedron (son) $\tau \in \mathcal{T}_j$ is covered by exactly one tetrahedron (father) $\tau' \in \mathcal{T}_{j-1}$, and any corner of τ is either a corner or an edge midpoint of τ'.*

Assumption 5.2. **Conformity:** *The intersection of any two tetrahedra $\tau, \tau' \in \mathcal{T}_j$ is either empty, a common vertex, a common edge or a common face.*

Assumption 5.3. **Nondegeneracy:** *The interior angles of all tetrahedra in the refinement sequence $\mathcal{T}_0, \mathcal{T}_1, \dots$ are bounded away from zero.*

A regular (red) refinement subdivides a tetrahedron τ into 8 equal volume subtetrahedra. We connect the edges of each face as in 2D regular refinement. We then cut off four subtetrahedra at the corners which are congruent to τ. An octahedron with three parallelograms remains in the interior. Cutting the octahedron along the two faces of these parallelograms, we obtain four more subtetrahedra which are not necessarily congruent to τ. We choose the diagonal of the parallelogram so that the successive refinements always preserve nondegeneracy [1, 4, 17, 22].

If a tetrahedron is marked for regular refinement, the resulting triangulation violates conformity A.5.2. Nonconformity is then remedied by irregular (green) refinement. In 3D, there are altogether $2^6 = 64$ possible edge refinements, of which 62 are irregular. One must pay extra attention to irregular refinement in the implementation due to the large number of possible nonconforming configurations. Bey [4] gives a methodical way of handling irregular cases. Using symmetry arguments, the 62 irregular cases can be divided into 9 different types. To ensure that the interior angles remain bounded away from zero, we enforce the following additional conditions. (Identical assumptions were made in [11] for their 2D refinement analogue.)

Assumption 5.4. *Irregular tetrahedra are not refined further.*

Assumption 5.5. *Only tetrahedra $\tau \in \mathcal{T}_j$ with $L(\tau) = j$ are refined for the construction of $\mathcal{T}_{j+1}$, where $L(\tau) = \min\{j : \tau \in \mathcal{T}_j\}$ denotes the level of τ.*

One should note that the restrictive character of A.5.4 and A.5.5 can be eliminated by a modification on the sequence of the tetrahedralizations [4]. On the other hand, it is straightforward to enforce both assumptions in a typical local refinement algorithm by minor modifications of the supporting datastructures for tetrahedral elements (cf. [14]). In any event, the proof technique (see (8.8) and (8.9)) requires both assumptions hold. The last refinement condition enforced for the possible 62 irregularly refined tetrahedra is stated as the following.

Assumption 5.6. *If three or more edges are refined and do not belong to a common face, then the tetrahedron is refined regularly.*

We note that the d-vertex adjacency generation bound for simplices in $\mathbb{R}^d$ which are adjacent at d vertices *is the primary result required in the support of a basis function so that (7.1) holds, and depends delicately on the particular details of the local refinement procedure rather than on shape regularity of the elements.* The generation bound for simplices which are adjacent at $d-1, d-2, \dots$ vertices follows by using the shape regularity and the generation bound established for d-vertex adjacency. We provide rigorous generation bound proofs for all the adjacency types mentioned in the lemmas to follow when $d = 3$. The 2D version appeared in [11]; the 3D extension is as described below without any additional framework.

Lemma 5.1. *Let τ and τ' be two tetrahedra in $\mathcal{T}_j$ sharing a common face f. Then*

$$|L(\tau) - L(\tau')| \leq 1. \tag{5.1}$$

Proof. If $L(\tau) = L(\tau')$, then $0 \leq 1$, there is nothing to show. Without loss of generality, assume that $L(\tau) < L(\tau')$. Proof requires a detailed and systematic analysis. To show the line of reasoning, we first list the facts used in the proof:

1. $L(\tau') \leq j$ because by assumption $\tau' \in \mathcal{T}_j$. Then, $L(\tau) < j$.
2. By assumption $\tau \in \mathcal{T}_j$, meaning that τ was never refined from the level it was born $L(\tau)$ to level j.
3. Let τ'' be the father of τ'. Then $L(\tau'') = L(\tau') - 1 < j$.
4. $L(\tau) < L(\tau')$ by assumption, implying $L(\tau) \leq L(\tau'')$.
5. By (2), τ belongs to all the triangulations from $L(\tau)$ to j, in particular $\tau \in \mathcal{T}_{L(\tau'')}$, where by (3) $L(\tau'') < j$.

f is the common face of τ and τ' on level j. If τ' is obtained by regular refinement of its father τ'', then f is still the common face of τ and τ''. By (5) both $\tau,\ \tau'' \in \mathcal{T}_{L(\tau'')}$. Then, A.5.2 implies that f is the common face of τ and τ''. Hence, τ' must have been irregular.

On the other hand, $L(\tau) \leq L(\tau') - 1 = L(\tau'')$. Next, we proceed by eliminating the possibility that $L(\tau) < L(\tau'')$. If so, we repeat the above reasoning, and τ'' becomes irregular. τ'' is already the father of the irregular τ', contradicting A.5.4 for level $L(\tau'')$. Hence $L(\tau) = L(\tau'') = L(\tau') - 1$ concludes the proof. ∎

By A.5.4 and A.5.5, every tetrahedron at any $\mathcal{T}_j$ is geometrically similar to some tetrahedron in $\mathcal{T}_0$ or to a tetrahedron arising from an irregular refinement of some tetrahedron in $\mathcal{T}_0$. Then, there exist absolute constants $c_1,\ c_2$ such that

$$c_1 \ \mathrm{diam}(\bar{\tau}) \ 2^{-L(\tau)} \leq \mathrm{diam}(\tau) \leq c_2 \ \mathrm{diam}(\bar{\tau}) \ 2^{-L(\tau)}, \tag{5.2}$$

where $\bar{\tau}$ is the father of τ in the initial mesh.

LEMMA 5.2. *Let τ and τ' be two tetrahedra in $\mathcal{T}_j$ sharing a common edge (two vertices). Then there exists a finite number E depending on the shape regularity such that*

$$|L(\tau) - L(\tau')| \leq E. \tag{5.3}$$

Proof. Start with $\tau = \tau_1$, obtain the face-adjacent neighbor τ_2 (either of the two faces), and then obtain the face-adjacent tetrahedron τ_3 of τ_2, repeat this process δ times until you reach $\tau' = \tau_\delta$. After you pick one of the two faces, the direction of the face-adjacent tetrahedra is determined. δ is always a finite number due to shape regularity. Lemma follows by face-adjacent neighbor relation (5.1). ∎

LEMMA 5.3. *Let τ and τ' be two tetrahedra in $\mathcal{T}_j$ sharing a common vertex. Then there exists a finite number V depending on the shape regularity such that*

$$|L(\tau) - L(\tau')| \leq V. \tag{5.4}$$

Proof. Take one edge from τ and τ' where the edges meet at the common vertex. By shape regularity, there exist η (a bounded number) many edge-adjacent tetrahedra between $\tau_1 = \tau$ and $\tau_\eta = \tau'$. By the above construction in Lemma 5.2, there exist $\delta_{1,2}$ many face-adjacent tetrahedra between τ_1 and τ_2. We repeat this process until we place $\delta_{\eta-1,\eta}$ many tetrahedra between $\tau_{\eta-1}$ and τ_η. Hence, there exist $\sum_{i=1}^{\eta-1} \delta_{i,i+1}$ face-adjacent tetrahedra between τ and τ'. Again, face-adjacent neighbor relation (5.1) concludes the lemma with $V = \sum_{i=1}^{\eta-1} \delta_{i,i+1}$. ∎

Consequently, simplices in the support of a basis function are comparable in size as indicated in (5.5). This is usually called *patchwise quasiuniformity*. Furthermore, it was shown in [1] that patchwise quasiuniformity (5.5) holds for 3D marked tetrahedron bisection by Joe and Liu [15] and for 2D newest vertex bisection by Sewell [20] and Mitchell [16]. Due to restrictive nature of the proof technique (see (8.8) and (8.9)), we focus on refinement procedures which obey A.5.4 and A.5.5.

LEMMA 5.4. *There is a constant depending on the shape regularity of $\mathcal{T}_j$ and the quasiuniformity of $\mathcal{T}_0$, such that*

$$\frac{\operatorname{diam}(\tau)}{\operatorname{diam}(\tau')} \le c, \quad \forall \tau, \tau' \in \mathcal{T}_j, \quad \tau \cap \tau' \neq \emptyset. \tag{5.5}$$

Proof. τ and τ' are either face-adjacent (d vertices), edge-adjacent ($d-1$ vertices), or vertex-adjacent, and are handled by (5.1), (5.3), (5.4), respectively.

$$\begin{aligned}\frac{\operatorname{diam}(\tau)}{\operatorname{diam}(\tau')} &\le c\; 2^{|L(\tau)-L(\tau')|}\;\frac{\operatorname{diam}(\bar\tau)}{\operatorname{diam}(\bar{\tau'})} \quad \text{(by (5.2))}\\ &\le c\; 2^{\max\{1,E,V\}}\;\gamma^{(0)} \quad \text{(by (5.1), (5.3), (5.4) and quasiuniformity of } \mathcal{T}_0)\end{aligned}$$

□

6. Local smoothing computational complexity. The following result from [5] establishes a bound for the number of nodes used for smoothing (those created in the refinement region by the BEK procedure) so that the BPX preconditioner has provably optimal (linear) computational complexity per iteration.

LEMMA 6.1. *The total number of nodes used for smoothing satisfies the bound:*

$$\sum_{j=0}^{J} \tilde N_j \le \frac{5}{3} N_J - \frac{2}{3} N_0. \tag{6.1}$$

Proof. See [5]. □

A similar result for 2D red-green refinement was given by Oswald (see page 95 in [19]). In the general case of local smoothing operators which involve smoothing over newly created basis functions plus some additional set of local neighboring basis functions, one can extend the arguments from [5] and [19] using shape regularity.

7. Establishing optimality of the BPX preconditioner. In this section, we extend the Dahmen-Kunoth framework of to three spatial dimensions and the extension closely follows the original work. However, the general case for $d \ge 1$ spatial dimensions is not in the literature, and therefore we present it below.

Our motivation is to form a stable basis in the following sense [19].

$$\| \sum_{x_i \in \mathcal{N}_j} u_i \phi_i^{(j)} \|_{L_2(\Omega)} \eqsim \| \{\text{volume}^{1/2}(\text{supp } \phi_i^{(j)})\; u_i\}_{x_i \in \mathcal{N}_j} \|_{l_2}. \tag{7.1}$$

The basis stability (7.1) will then guarantee that the Bernstein estimate (4.1) holds, which is the first step in establishing the norm equivalence. For a stable basis, functions with small supports have to be augmented by an appropriate scaling so that $\|\phi_i^{(j)}\|_{L_2(\Omega)}$ remains roughly the same for all basis functions. This is reflected in volume(supp $\phi_i^{(j)}$) by defining:

$$L_{j,i} = \min\{L(\tau) : \tau \in \mathcal{T}_j,\; x_i \in \tau\}. \tag{7.2}$$

Then

$$\text{volume}(\text{supp } \phi_i^{(j)}) \eqsim 2^{-dL_{j,i}}.$$

We prefer to use an equivalent notion of basis stability; a basis is called L_2-*stable Riesz basis* (cf [3]) if:

$$\| \sum_{x_i \in \mathcal{N}_j} \hat{u}_i \hat{\phi}_i^{(j)} \|_{L_2(\Omega)} \eqsim \| \{\hat{u}_i\}_{x_i \in \mathcal{N}_j} \|_{l_2}, \tag{7.3}$$

where $\hat{\phi}_i^{(j)}$ denotes the scaled basis, and the relationship between (7.1) and (7.3) is given as follows:

$$\hat{\phi}_i^{(j)} = 2^{d/2 L_{j,i}} \ \phi_i^{(j)}, \quad \hat{u}_i = 2^{-d/2 L_{j,i}} \ u_i, \quad x_i \in \mathcal{N}_j. \tag{7.4}$$

Remark 7.1. *Our construction works for any d-dimensional setting with the scaling (7.4). However, it is not clear how to define face-adjacency relations for $d > 3$. If such relations can be defined through some topological or geometrical abstraction, then our framework naturally extends to d-dimensional local refinement strategies, and hence the optimality of the BPX preconditioner can be guaranteed in $\mathbb{R}^d$, $d \geq 1$. The analysis is done purely with basis functions, completely independent of the underlying mesh geometry.*

For linear g, the element mass matrix gives rise to the following useful formula.

$$\|g\|^2_{L_2(\tau)} = \frac{\text{volume}(\tau)}{(d+1)(d+2)} \ (\sum_{i=1}^{d+1} g(x_i)^2 + [\sum_{i=1}^{d+1} g(x_i)]^2), \tag{7.5}$$

where, $i = 1, \ldots, d+1$ and x_i is a vertex of τ, $d = 2, 3$. In view of (7.5), we have that

$$\|\hat{\phi}_i^{(j)}\|^2_{L_2(\Omega)} = 2^{dL_{j,i}} \ \frac{\text{volume}(\text{supp } \hat{\phi}_i^{(j)})}{(d+1)(d+2)}.$$

Since the min in (7.2) is attained, there exists at least one $\tau \in$ supp $\hat{\phi}_i^{(j)}$ such that $L(\tau) = L_{j,i}$. By (5.2) we have

$$2^{L_{j,i}} \eqsim \frac{\text{diam}(\tau)}{\text{diam}(\bar{\tau})}. \tag{7.6}$$

Also,

$$\text{volume}(\text{supp } \hat{\phi}_i^{(j)}) \eqsim \sum_{i=1}^{M} \text{diam}^3(\tau_i), \quad \tau_i \in \text{supp } \hat{\phi}_i^{(j)}. \tag{7.7}$$

By (5.5), we have

$$\text{diam}(\tau_i) \eqsim \text{diam}(\tau). \tag{7.8}$$

Combining (7.7) and (7.8), we conclude

$$\text{volume}(\text{supp } \hat{\phi}_i^{(j)}) \eqsim M\text{diam}^3(\tau). \tag{7.9}$$

Finally then, (7.6) and (7.9) yield

$$2^{dL_{j,i}}\text{volume}(\text{supp }\hat{\phi}_i^{(j)}) \eqsim M\ \frac{1}{\text{diam}^3(\bar{\tau})}.$$

M is a uniformly bounded constant by shape regularity. One can view the size of any tetrahedron in $\mathcal{T}_0$, in particular size of $\bar{\tau}$, as a constant. The reason is the following: A.5.4 and A.5.5 force every tetrahedron at any $\mathcal{T}_j$ to be geometrically similar to some tetrahedron in $\mathcal{T}_0$ or to a tetrahedron arising from an irregular refinement of some tetrahedron in $\mathcal{T}_0$, hence, to some tetrahedron of a fixed finite collection. Combining the two arguments above, we have established that

$$\|\hat{\phi}_i^{(j)}\|_{L_2(\Omega)} \eqsim 1, \quad x_i \in \mathcal{N}_j. \tag{7.10}$$

Let $g = \sum_{x_i\in\mathcal{N}_j} \hat{u}_i\hat{\phi}_i^{(j)} \in \mathcal{S}_j$. For any $\tau \in \mathcal{T}_j$ we have that

$$\|g\|^2_{L_2(\tau)} \le c \sum_{x_i\in\mathcal{N}_{j,\tau}} |\hat{u}_i|^2\|\hat{\phi}_i^{(j)}\|^2_{L_2(\Omega)}, \tag{7.11}$$

where $\mathcal{N}_{j,\tau} = \{x_i \in \mathcal{N}_j : x_i \in \tau\}$, which is uniformly bounded in $\tau \in \mathcal{T}_j$ and $j \in \mathbb{N}_0$. By the scaling (7.4), we get equality in the estimate below. The inequality is a standard inverse estimate where one bounds $g(x_i)$ using formula (7.5) and by handling the volume in the formula by (5.2):

$$|\hat{u}_i|^2 = 2^{-dL_{j,i}}|g(x_i)|^2 \le c\ 2^{-dL_{j,i}}2^{dL_{j,i}}\|g\|^2_{L_2(\tau)}. \tag{7.12}$$

We sum up over $\tau \in \mathcal{T}_j$ in (7.11) and (7.12), and by using (7.10) we achieve stability (7.3). This allows us to establish the Bernstein estimate (4.1).

LEMMA 7.1. *For the scaled basis (7.4), the Bernstein estimate (4.1) holds for* $\beta = 3/2$

Proof. (7.10) with (7.11) and (7.12) assert that the scaled basis (7.4) is stable in the sense of (7.3). Hence, (4.1) holds by Theorem 4 in [19]. Note that the proof actually works independently of the spatial dimension. □

8. Lower bound in the norm equivalence. The Jackson estimate for Besov spaces is defined as follows:

$$\inf_{g\in\mathcal{S}_J} \|f-g\|_{L_p} \le c\ \omega_\alpha(f, 2^{-J})_p, \quad f \in L_p(\Omega), \tag{8.1}$$

where c is a constant independent of f and J, and α is an integer. In the uniform refinement setting, (8.1) is used to obtain the lower bound. However, in local the refinement setting, (8.1) holds only for functions whose singularities are somehow well-captured by the mesh geometry. For instance, if a mesh is designed to pick up the singularity at $x = 0$ of $y = 1/x$, then on the same mesh we will not be able to recover a singularity at $x = 1$ of $y = 1/(x-1)$. Hence the Jackson estimate (8.1) cannot hold in a general setting, i.e. for $f \in W_p^k$. In order to get the lower bound, we focus on estimating v_J directly, as in [11] for the 2D setting.

To begin we borrow the quasi-interpolant construction from [11]. Let $\tau \in \mathcal{T}_j$ be a tetrahedron with vertices x_1, x_2, x_3, x_4. Clearly the restrictions of $\hat{\phi}_i^{(j)}$ to τ are linearly independent over τ where $x_i \in \{x_1, x_2, x_3, x_4\}$. Then, there exists a unique set of linear polynomials $\psi_1^\tau, \psi_2^\tau, \psi_3^\tau, \psi_4^\tau$ such that

$$\int_\tau \hat{\phi}_k^{(j)}(x,y,z)\psi_l^\tau(x,y,z)dxdydz = \delta_{kl}, \quad x_k, x_l \in \{x_1, x_2, x_3, x_4\}. \tag{8.2}$$

For $x_i \in \mathcal{N}_j$ and $\tau \in \mathcal{T}_j$, define a function for $x_i \in \tau$

$$\xi_i^{(j)}(x,y,z) = \begin{cases} \frac{1}{M_i}\psi_i^{\tau}(x,y,z), & (x,y,z) \in \tau \\ 0, & (x,y,z) \notin \text{supp } \hat{\phi}_i^{(j)} \end{cases}, \tag{8.3}$$

where M_i is the number of tetrahedra in $\mathcal{T}_j$ in supp $\hat{\phi}_i^{(j)}$. By (8.2) and (8.3), we obtain

$$(\xi_k^{(j)}, \hat{\phi}_l^{(j)}) = \int_{\Omega} \xi_k^{(j)}(x,y,z)\hat{\phi}_l(x,y,z)dxdydz = \delta_{kl}, \quad x_k, x_l \in \mathcal{N}_j. \tag{8.4}$$

We can now define a quasi-interpolant, in fact a *projection* onto $\mathcal{S}_j$, such that

$$(\tilde{Q}_j f)(x,y,z) = \sum_{x_i \in \mathcal{N}_j} (f, \xi_i^{(j)})\hat{\phi}_i^{(j)}(x,y,z). \tag{8.5}$$

As remarked earlier, due to (8.3) the slice operator term $\tilde{Q}_j - \tilde{Q}_{j-1}$ will vanish outside the refined set Ω_j defined in (2.1), which is critical for enforcing optimal (linear) complexity of the smoother. One can easily observe by (7.10) and (8.4) that

$$\|\xi_i^{(j)}\|_{L_2(\Omega)} \eqsim 1, \quad x_i \in \mathcal{N}_j, \ j \in \mathbb{N}_0. \tag{8.6}$$

Letting $\Omega_{j,\tau} = \bigcup\{\tau' \in \mathcal{T}_j : \ \tau \cap \tau' \neq \emptyset\}$, we can conclude from (7.10) and (8.6) that

$$\|\tilde{Q}_j f\|_{L_2(\tau)} = \| \sum_{x_k \in \mathcal{N}_{j,\tau}} (f, \xi_l^{(j)})\hat{\phi}_k^{(j)} \|_{L_2(\tau)} \leq c\|f\|_{L_2(\Omega_{j,\tau})}. \tag{8.7}$$

We define now a subset of the triangulation where the refinement activity stops, meaning that all tetrahedra in $\mathcal{T}_j^*, \ j \leq m$ also belong to $\mathcal{T}_m$:

$$\mathcal{T}_j^* = \{\tau \in \mathcal{T}_j : \ L(\tau) < j, \ \Omega_{j,\tau} \cap \tau' = \emptyset, \ \forall \tau' \in \mathcal{T}_j \text{ with } L(\tau') = j\}. \tag{8.8}$$

Due to the local support of the dual basis functions $\xi_i^{(j)}$ and the fact that $\tilde{Q}_j$ is a projection, one gets

$$\|g - \tilde{Q}_j g\|_{L_2(\tau)} = 0, \quad \tau \in \mathcal{T}_j^*. \tag{8.9}$$

Since $\tilde{Q}_j$ is a projection onto linear finite element space, it fixes polynomials of degree at most 1 (i.e. $\Pi_1(\mathbb{R}^3)$). Using this fact and (8.7), we arrive:

$$\begin{aligned} \|g - \tilde{Q}_j g\|_{L_2(\tau)} &\leq \|g - P\|_{L_2(\tau)} + \|\tilde{Q}_j(P - g)\|_{L_2(\tau)} \\ &\leq c \ \|g - P\|_{L_2(\Omega_{j,\tau})}, \quad \tau \in \mathcal{T}_j \setminus \mathcal{T}_j^*. \end{aligned} \tag{8.10}$$

We would like to bound the right hand side of (8.10) in terms of a modulus of smoothness in order to reach a Jackson-type estimate. Following [11], we utilize a modified modulus of smoothness:

$$\tilde{\omega}_k(f,t,\Omega)_p^p = t^{-s} \int_{[-t,t]^s} \|\Delta_h^k f\|_{L_p(\Omega_{k,h})}^p \ dh.$$

They can be shown to be equivalent:

$$\tilde{\omega}_{k+1}(f,t,\Omega)_p \eqsim \omega_{k+1}(f,t,\Omega)_p.$$

The equivalence in the one-dimensional setting can be found in Lemma 5.1 in [12].

For τ a simplex in $\mathbb{R}^d$ and $t = \operatorname{diam}(\tau)$, a Whitney estimate shows that [13, 18, 21]

$$\inf_{P \in \Pi_k(\mathbb{R}^d)} \|f - P\|_{L_p(\tau)} \le c\tilde{\omega}_{k+1}(f,t,\tau)_p, \tag{8.11}$$

where c depends only on the smallest angle of τ but not on f and t. The reason why $\tilde{Q}_j$ works well for tetrahedralization in 3D is the fact that the Whitney estimate (8.11) remains valid for any spatial dimension. $\mathcal{T}_j \setminus \mathcal{T}_j^*$ is the part of the tetrahedralization $\mathcal{T}_j$ where refinement is active at every level. Then, in view of (5.5)

$$\operatorname{diam}(\Omega_{j,\tau}) \eqsim 2^{-j}, \quad \tau \in \mathcal{T}_j \setminus \mathcal{T}_j^*.$$

Taking the inf over $P \in \Pi_1(\mathbb{R}^3)$ in (8.10) and using the Whitney estimate (8.11) we conclude

$$\|g - \tilde{Q}_j g\|_{L_2(\tau)} \le c\tilde{\omega}_2(g, 2^{-j}, \Omega_{j,\tau})_2.$$

Recalling (8.9) and summing over $\tau \in \mathcal{T}_j \setminus \mathcal{T}_j^*$ gives rise to

$$\|g - \tilde{Q}_j g\|_{L_2(\Omega)} \le c\tilde{\omega}_2(g, 2^{-j}, \Omega)_2 \le \tilde{c}\ \omega_2(g, 2^{-j}, \Omega)_2,$$

where we have switched from the modified modulus of smoothness to the standard one. With (4.2) one then has

$$v_J = O(1), \quad J \to \infty. \tag{8.12}$$

9. Conclusion. In this article, we examined the Bramble-Pasciak-Xu (BPX) preconditioner in the setting of local 3D mesh refinement. In particular, we extended the 2D optimality result for BPX due to Dahmen and Kunoth to the local 3D red-green refinement procedure introduced by Bornemann-Erdmann-Kornhuber (BEK). The extension involved establishing that the locally enriched finite element subspaces produced by the BEK procedure allow for the construction of a scaled basis which is formally Riesz stable. This in turn rested entirely on establishing a number of geometrical relationships between neighboring simplices produced by the local refinement algorithms. We remark again that shape regularity of the elements produced by the refinement procedure is insufficient to construct a stable Riesz basis for finite element spaces on locally adapted meshes. The d-vertex adjacency generation bound for simplices in $\mathbb{R}^d$ is the primary result required to establish patchwise quasiuniformity for stable Riesz basis construction, and this result depends delicately on the particular details of the local refinement procedure rather than on shape regularity of the elements. We also noted in §5 that these geometrical properties have been established in [1] for purely bisection-based refinement procedures that have been shown to be asymptotically non-degenerate, and therefore also allow for the construction of a stable Riesz basis.

To address the practical computational complexity of an implementable version of BPX, we showed that the number of degrees of freedom used for smoothing is bounded by a constant times the number of degrees of freedom introduced at that level of refinement. This indicates that a practical implementable version of the BPX preconditioner for the local 3D refinement setting considered here has provably optimal (linear) computational complexity per iteration, as well as having a uniformly bounded condition number.

The theoretical framework established here supports arbitrary spatial dimension $d \geq 1$; we indicated clearly which geometrical properties must be re-established to show BPX optimality for spatial dimension $d \geq 4$. All of the results contained in this article require no smoothness assumptions on the PDE coefficients beyond those required for well-posedness in H^1.

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