arXiv:1108.3660v3 [math.NA] 16 Nov 2011

CONVERGENCE OF GOAL-ORIENTED ADAPTIVE FINITE ELEMENT METHODS FOR NONSYMMETRIC PROBLEMS

MICHAEL HOLST AND SARA POLLOCK

ABSTRACT. In this article we develop convergence theory for a class of goal-oriented adaptive finite element algorithms for second order nonsymmetric linear elliptic equations. In particular, we establish contraction and quasi-optimality results for a method of this type for second order Dirichlet problems involving the elliptic operator $\mathcal{L}u = \nabla \cdot (A\nabla u) - b \cdot \nabla u - cu$, with A Lipschitz, almost-everywhere symmetric positive definite (SPD), with b divergence-free, and with $c \geq 0$. We first describe the problem class and review some standard facts concerning conforming finite element discretization and error-estimate-driven adaptive finite element methods (AFEM). We then describe a goal-oriented variation of standard AFEM (GOAFEM). Following the recent work of Mommer and Stevenson for symmetric problems, we establish contraction of GOAFEM. We also then show convergence in the sense of the goal function. Our analysis approach is signficantly different from that of Mommer and Stevenson, combining the recent contraction frameworks developed by Cascon et. al, by Nochetto, Siebert, and Veeser, and by Holst, Tsogtgerel, and Zhu. In the last part of the paper we perform a complexity analysis, and establish quasi-optimal cardinality of GOAFEM. We include an appendix discussion of the duality estimate as we use it here in an effort to make the paper more self-contained.

CONTENTS

Date: November 17, 2011.

Key words and phrases. Adaptive methods, elliptic equations, non-symmetric problems, quasi-orthogonality, duality, approximation theory, residual-based error estimator, convergence, contraction, optimality, *a priori* estimates, *a posteriori* estimates, goal oriented.

MH was supported in part by NSF Awards 0715146 and 0915220, and by DOD/DTRA Award HDTRA-09-1-0036.

SP was supported in part by NSF Award 0715146.

1. INTRODUCTION

In this article we develop convergence theory for a class of goal-oriented adaptive finite element methods for second order nonsymmetric linear elliptic equations. In particular, we report contraction and quasi-optimality results for a method of this type for the problem

$$-\nabla \cdot (A\nabla u) + b \cdot \nabla u + cu = f, \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0, \quad \text{on } \partial\Omega, \tag{1.2}$$

with $\Omega \subset \mathbb{R}^d$ a polyhedral domain, $d = 2$ or 3, with A Lipschitz, almost-everywhere (a.e.) symmetric positive definite (SPD), with b divergence-free, and with $c \geq 0$. The standard weak formulation of this problem reads: Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = f(v), \quad \forall v \in H_0^1(\Omega), \tag{1.3}$$

where

$$a(u, v) = \int_\Omega A\nabla u \cdot \nabla v + b \cdot \nabla uv + cuv \, dx, \qquad f(v) = \int_\Omega fv \, dx. \tag{1.4}$$

Our approach is to first describe the problem class in some detail, and review some standard facts concerning conforming finite element discretization and error-estimate-driven adaptive finite element methods (AFEM). We will then describe a goal-oriented variation of standard AFEM (GOAFEM). Following the recent work of Mommer and Stevenson [10] for symmetric problems, we establish contraction of GOAFEM. We also show convergence in the sense of the goal function. Our analysis approach is signficantly different from that of Mommer and Stevenson [10], combining the recent contraction frameworks of Cascon et. al [4], of Nochetto, Siebert, and Veeser [11], and of Holst, Tsogtgerel, and Zhu [8]. We also give a complexity analysis, and establish quasi-optimal cardinality of GOAFEM.

The goal-oriented problem concerns achieving a target quality in a given linear functional $g\colon H_0^1(\Omega) \to \mathbb{R}$ of the weak solution $u \in H_0^1(\Omega)$ of the problem (1.3). For example, $g(u) = \int_\Omega \frac{1}{|\omega|}\chi_\omega u$, the average value of u over some domain $\omega \subset \Omega$. By writing down the adjoint operator, $a^*(z, v) = a(v, z)$, we consider the *adjoint* or *dual* problem: find $z \in H_0^1(\Omega)$ such that $a^*(z, v) = g(v),$ for all $v \in H_0^1(\Omega)$. It has been shown for the symmetric form ($b = 0$) of problem (1.1)–(1.2) with piecewise constant SPD diffusion cofficient A (and with $c = 0$), that by solving the *primal* and *dual* problems simultaneously, one may converge to an approximation of $g(u)$ faster than by approximating u then $g(u)$, when forcing contraction in only the primal problem [10]. We will follow the same general approach in order to establish similar goal-oriented AFEM results for nonsymmetric problems. However, in order to handle nonsymmetry, we will follow the technical approach in [9, 4, 8], and rely largely on establishing quasi-orthogonality. In particilar, contraction results are established in [9, 4] for (1.1)–(1.2) in the case that A is SPD, Lipschitz or piecewise Lipschitz, b is divergence-free, and $c \geq 0$. In [8], quasi-orthogonality is used as the basis for establishing contraction of AFEM for two classes of nonlinear problems. As in these earlier efforts, relying on quasi-orthogonality will require that we assume that the initial mesh is sufficiently fine, and that the solution to the dual problem $a^*(w, v) = g(v),\ g \in L_2(\Omega)$ is sufficiently smooth, e.g. in $H^2_{\text{loc}}(\Omega)$.

Following [8], the contraction argument developed in this paper will follow from first establishing three preliminary results for two successive AFEM approximations u_1 and u_2, and then applying the Dörfler marking strategy:

1) Quasi-orthogonality (§3.1): There exists $\Lambda > 1$ such that
$$|||u - u_2|||^2 \le \Lambda |||u - u_1|||^2 - |||u_2 - u_1|||^2.$$
2) Error estimator as upper bound on error (§3.2): There exists $C_1 > 0$ such that
$$|||u - u_k|||^2 \le C_1 \eta_k^2(u_k, \mathcal{T}_k), \quad k = 1, 2.$$
3) Estimator reduction (§3.4): For $\mathcal{M}$ the marked set that takes refinement $\mathcal{T}_1 \to \mathcal{T}_2$, for positive constants $\lambda < 1$ and Λ_1 and any $\delta > 0$
$$\eta_2^2(v_2, \mathcal{T}_2) \le (1+\delta)\{\eta_1^2(v_1, \mathcal{T}_1) - \lambda \eta_1^2(v_1, \mathcal{M}\} + (1+\delta^{-1})\Lambda_1 \eta_0^2 |||v_2 - v_1|||.$$

The marking strategy used is the original Dörfler strategy; elements are marked for refinement based on indicators alone. The marked set $\mathcal{M}$ must satisfy
$$\sum_{T \in \mathcal{M}} \eta_k^2(u_k, T) \ge \theta^2 \eta_k^2(u_k, \mathcal{T}_k).$$
In the goal-oriented method, a second marked set is chosen based on an error indicator for the dual problem associated with the given goal functional, and the union of the two marked sets is then used for refinement.

A main advantage of the approach in [4] is that it does not require an interior node property. This allows us to establish the necessary results for contraction without taking full refinements of the mesh at each iteration. This improvement follows from the use of the local perturbation estimate or local Lipschitz property rather than the estimator as lower bound on error. We use the standard lower bound estimate as found in [9] for optimality arguments in the second part of the paper concerning quasi-optimality of the method.

There are three main notions of error used throughout this paper. The energy error $|||u - u_k|||$, the quasi-error and the total-error. The *energy error* is defined by the symmetric part of the bilinear form that arises from the given differential operator in (1.3). The *quasi-error* is the l_2 sum of the energy-error and scaled error estimator
$$Q_k(u_k, \mathcal{T}_k) := (|||u - u_k|||^2 + \gamma \eta_k^2)^{1/2},$$
and this is the quantity that is reduced at each iteration of the algorithm. In §3 the quasi-error is shown to satisfy
$$|||u - u_{k+1}|||^2 + \gamma \eta_{k+1}^2 \le \alpha^2 \left(|||u - u_k|||^2 + \gamma \eta_k^2 \right), \ \alpha < 1.$$
The *total error* includes the oscillation term rather than the estimator
$$E_k(u_k, \mathcal{T}_k) := (|||u - u_k|||^2 + \text{osc}_k^2)^{1/2}.$$
The oscillation term captures the higher-frequency oscillations in the residual missed by the averaging of the finite element method. While the quasi-error is the focus of the contraction arguments, it is the total error that will be critical to complexity analysis. Therefore, we will need to establish various preliminary results for both types of error.

The quasi-optimality of the goal oriented method in §4 is developed with respect to the total error which is shown to satisfy Cea's lemma. The cardinality result
$$\#\mathcal{T}_k - \#\mathcal{T}_0 \le S(\theta) \left\{ M_p \left(1 + \frac{\gamma_p}{c_2}\right)^{1/2s} Q_k^{-1/s}(u_k, \mathcal{T}_k) \right.$$
$$\left. + M_d \left(1 + \frac{\gamma_d}{c_2}\right)^{1/2t} Q_k^{-1/t}(z_k, \mathcal{T}_k) \right\}$$

bounds the growth of the adaptive mesh with respect to the quasi-error of both problems. An equivalence between the quasi-error and total error is established in §4.

A final brief comment is in order concerning the notation used here compared to that in [4] and the related literature. In [4], the number of times each marked element is refined is denoted b. In this article, each marked element is refined once. Therefore, b will be reserved for the convection term in the nonsymmetric problem. The constant C will denote a generic but global constant that may depend on the data and the condition of the initial mesh $\mathcal{T}_0$, and may change from step to step.

Outline of the paper. The remainder of the paper is structured as follows. In §2, we first describe the problem class and review some standard facts concerning conforming finite element discretization and error-estimate-driven adaptive finite element methods (AFEM). In §2.3, we then describe a goal-oriented variation of the standard approach to AFEM (GOAFEM). Following the recent work of Mommer and Stevenson for symmetric problems, in §3 we establish contraction of goal-oriented AFEM. We also then show convergence in §3.6 in the sense of the goal function. Our analysis approach is signficantly different, combining the recent contraction frameworks developed by Cascon et. al [4], Nochetto, Siebert, and Veeser [11], and by Holst, Tsogtgerel, and Zhu [8]. In §4, we consider complexity questions, and establish quasi-optimal cardinality of GOAFEM. We recap the results in §5, and point out some remaining open problems.

2. PROBLEM CLASS, DISCRETIZATION, GOAL-ORIENTED AFEM

2.1. **Problem class, weak formulation, spaces and norms.** Consider the nonsymmetric problem (1.3), where as in (1.4) we have

$$a(u,v) = \langle A\nabla u, \nabla v\rangle + \langle b\cdot\nabla u, v\rangle + \langle cu, v\rangle.$$

Here we have introduced the notation $\langle\cdot,\cdot\rangle$ for the L_2 inner-product over $\Omega\subset\mathbb{R}^d$. The adjoint or dual problem is: Find $z\in H_0^1(\Omega)$ such that

$$a^*(z,v) = g(v) \text{ for all } v\in H_0^1(\Omega) \tag{2.1}$$

where $a^*(\,\cdot\,,\,\cdot\,)$ is the formal adjoint of $a(\,\cdot\,,\,\cdot\,)$, and where the functional is defined through

$$g(u) = \int_\Omega gu\,dx, \tag{2.2}$$

for some given $g\in L_2(\Omega)$. We will make the following assumptions on the data:

Assumption 2.1 (Problem data)**.** *The problem data $D=(A,b,c,f)$ and dual problem data $D^*=(A,-b,c,g)$ satisfy*

1) $A:\overline{\Omega}\to\mathbb{R}^{d\times d}$, Lipschitz, and a.e. symmetric positive-definite:

$$\textit{ess inf}_{x\in\Omega}\lambda_{min}(A(x)) = \mu_0 > 0, \tag{2.3}$$

$$\textit{ess sup}_{x\in\Omega}\lambda_{max}(A(x)) = \mu_1 < \infty. \tag{2.4}$$

2) $b:\overline{\Omega}\to\mathbb{R}^d$, with $b_k\in L_\infty(\Omega)$, and b divergence-free.
3) $c:\overline{\Omega}\to\mathbb{R}$, with $c\in L_\infty(\Omega)$, and $c(x)\geq 0$ for all $x\in\Omega$.
4) $f,g\in L_2(\Omega)$.

The native norm is the Sobolev H^1 norm given by

$$\|v\|_{H^1}^2 = \langle\nabla v,\nabla v\rangle + \langle v,v\rangle. \tag{2.5}$$

The L_p norm of a vector valued function v over domain ω is defined here as the l_2 norm of the $L_p(\omega)$ norm of each component

$$\|v\|_{L_p(\omega)} = \left(\sum_{j=1}^{d} \left(\int_\omega v_j^p \right)^{2/p} \right)^{1/2}, \; p = 1, 2, \ldots$$
$$\|v\|_{L_\infty(\omega)} = \left(\sum_{j=1}^{d} \left(\operatorname*{ess\,sup}_\omega v_j \right)^2 \right)^{1/2}. \tag{2.6}$$

Similarly, the L_p norm of a matrix valued function M over domain ω is defined as the Frobenius norm of the $L_p(\omega)$ norm of each component

$$\|M\|_{L_p(\omega)} = \left(\sum_{i,j=1}^{d} \left(\int_\omega M_{ij}^p \right)^{2/p} \right)^{1/2}, \quad p = 1, 2, \ldots$$
$$\|M\|_{L_\infty(\omega)} = \left(\sum_{ij=1}^{d} \left(\operatorname*{ess\,sup}_\omega M_{ij} \right)^2 \right)^{1/2}. \tag{2.7}$$

We note that one could employ other equivalent discrete l_p norms in the definitions (2.6) and (2.7), however this choice simplifies the analysis.

Continuity of $a(\,\cdot\,,\,\cdot\,)$ follows from the Hölder inequality, and bounding the L_2 norm of the function and its gradient by the H^1 norm

$$a(u, v) \le (\mu_1 + \|b\|_{L_\infty} + \|c\|_{L_\infty}) \|u\|_{H^1} \|v\|_{H^1} = M_c \|u\|_{H^1} \|v\|_{H^1}. \tag{2.8}$$

Coercivity follows from the Poincaré inequality with constant C_Ω and the divergence-free condition

$$a(v, v) \ge \mu_0 |v|_{H^1}^2 \ge C_\Omega \mu_0 \|v\|_{H^1}^2 = m_{\mathcal{E}}^2 \|v\|_{H^1}^2, \tag{2.9}$$

where the coercivity constant $m_{\mathcal{E}}^2 \coloneqq C_\Omega \mu_o$. Continuity and coercivity imply existence and uniqueness of the solution by the Lax-Milgram Theorem [7]. The adjoint operator $a^*(\ ,\ )$ is given by

$$a^*(v, u) \coloneqq a(u, v), \qquad u, v \in H_0^1(\Omega).$$

Integration by parts on the convection term and the divergence-free condition imply

$$a^*(z, v) \coloneqq \langle A \nabla z, \nabla v \rangle - \langle b \cdot \nabla z, v \rangle + \langle cz, v \rangle. \tag{2.10}$$

Define the energy semi-norm by

$$|\!|\!| v |\!|\!|^2 \coloneqq a(v, v). \tag{2.11}$$

Non-negativity follows directly from the coercivity estimate (2.9)

$$|\!|\!| v |\!|\!|^2 \ge m_{\mathcal{E}}^2 \|v\|_{H^1}^2, \tag{2.12}$$

which establishes the energy semi-norm as a norm. Putting this together with the reverse inequality

$$|\!|\!| v |\!|\!|^2 \le \mu_1 |\nabla v|_{L_2}^2 + \|c\|_{L_\infty} \|v\|_{L_2}^2 \implies |\!|\!| v |\!|\!| \le M_{\mathcal{E}} \|v\|_{H^1}, \tag{2.13}$$

establishes the equivalence between the native and energy norms with the constant $M_{\mathcal{E}} = (\mu_1 + \|c\|_{L_\infty})^{1/2}$.

2.2. **Finite element approximation.** We employ a standard conforming piecewise polynomial finite element approximation below.

Assumption 2.2 (Finite element mesh)**.** *We make the following assumptions on the underlying simplex mesh:*

1) *The initial mesh $\mathcal{T}_0$ is conforming.*
2) *The mesh is refined by newest vertex bisection* [2], [10] *at each iteration.*
3) *The initial mesh $\mathcal{T}_0$ is sufficiently fine. In particular, it satisfies (3.6).*

Based on assumptions 2.2 we have the following mesh constants.

1) Define
$$h_{\mathcal{T}} \coloneqq \max_{T \in \mathcal{T}} h_T, \quad \text{where } h_T = |T|^{1/d}. \tag{2.14}$$
In particular, h_0 is the initial mesh diameter.
2) Define the mesh constant $\gamma_N = 2\gamma_r$ where
$$\gamma_r = \frac{h_0}{h_{min}} \quad \text{and } \ h_{min} = \min_{T \in \mathcal{T}_0} h_T$$
then for any two elements $T, \tilde{T}$ in the same generation
$$h_T \leq \gamma_r h_{\tilde{T}}$$
and as neighboring elements may differ by at most one generation for any two neighboring elements T and T'
$$h_T \leq 2\gamma_r h_{T'} = \gamma_N h_{T'} \tag{2.15}$$
3) The minimal angle condition satisfied by newest vertex bisection implies the mesh-size h_T is comparable to h_σ, the size of any true-hyperface σ of T. In particular, there is a constant $\bar{\gamma}$
$$\frac{h_\sigma}{h_T} \leq \bar{\gamma}^2 \text{ for all } T. \tag{2.16}$$

Let $\mathbb{T}$ the set of conforming meshes derived from the initial mesh $\mathcal{T}_0$. Define $\mathbb{T}_N \subset \mathbb{T}$ by
$$\mathbb{T}_N = \{\mathcal{T} \in \mathbb{T} \,\big|\, \#\mathcal{T} - \#\mathcal{T}_0 \leq N\}.$$
For a conforming mesh $\mathcal{T}_1$ with a conforming refinement $\mathcal{T}_2$ we say $\mathcal{T}_2 \geq \mathcal{T}_1$. The set of refined elements is given by
$$\mathcal{R}_{1\to 2} \coloneqq \mathcal{R}_{\mathcal{T}_1 \to \mathcal{T}_2} \coloneqq \mathcal{T}_1 \setminus (\mathcal{T}_2 \cap \mathcal{T}_1). \tag{2.17}$$
An *overlay* of two meshes $\mathcal{T}_1 \geq \mathcal{T}_0$ and $\mathcal{T}_2 \geq \mathcal{T}_0$ where $\mathcal{T}_2$ is not generally a refinement of $\mathcal{T}_1$ is given by
$$\mathcal{T}_1 \oplus \mathcal{T}_2 \coloneqq \{T \in \mathcal{T}_1 \big| T \subseteq T' \text{ for some } T' \in \mathcal{T}_2\} \cup \{T \in \mathcal{T}_2 \big| T \subseteq T' \text{ for some } T' \in \mathcal{T}_1\} \tag{2.18}$$
and is itself conforming. Define the finite element space
$$\mathbb{V}_{\mathcal{T}} \coloneqq H^1_0(\Omega) \cap \prod_{T \in \mathcal{T}} \mathbb{P}_n(T) \quad \text{and } \mathbb{V}_k \coloneqq \mathbb{V}_{\mathcal{T}_k}. \tag{2.19}$$
For subsets $\omega \subseteq \mathcal{T}$,
$$\mathbb{V}_{\mathcal{T}}(\omega) \coloneqq H^1_0(\Omega) \cap \prod_{T \in \omega} \mathbb{P}_n(T), \tag{2.20}$$

where $\mathbb{P}_n(T)$ is the space of polynomials degree degree n over T. Denote the patch about $T \in \mathcal{T}$

$$\omega_T := T \cup \{T' \in \mathcal{T} \mid T \cap T' \text{ is a true-hyperface of } T\}. \tag{2.21}$$

For a d-simplex T, an true-hyperface is a $d-1$ dimensional face of T, *e.g.,* a face in 3D or an edge in 2D. Define the discrete primal problem: Find $u_k \in \mathbb{V}_k$ such that

$$a(u_k, v_k) = f(v_k),\ v_k \in \mathbb{V}_k, \tag{2.22}$$

and the discrete dual problem

$$a^*(z_k, v_k) = g(v_k),\ v_k \in \mathbb{V}_k. \tag{2.23}$$

2.3. **Goal oriented AFEM (GOAFEM).** As in [10] the goal oriented adaptive finite element method (GOAFEM) is based on the standard AFEM algorithm:

$$\text{SOLVE} \ \rightarrow\ \text{ESTIMATE} \ \rightarrow\ \text{MARK} \ \rightarrow\ \text{REFINE}\ .$$

In the goal oriented method, one enforces contraction of the quasi-error in both the primal problem and an associated dual problem. As shown in section §3.6, the error in the goal-function satisfies the bound

$$|g(u) - g(u_k)| = |a(u - u_k, z - z_k)| \le 2|\!|\!| u - u_k |\!|\!| \, |\!|\!| z - z_k |\!|\!|.$$

This motivates driving down the energy-error in both the primal and dual problems at each iteration. As noted in [4] the residual-based error estimator does not exhibit monotone behavior in general, although it is monotone non-increasing with respect to nested mesh refinement when applied to the same (coarse) function. The quasi-error is shown to contract for each problem for which mesh refinement satisfies the Dörfler property. However, refining the mesh with respect to the primal problem does not guarantee the quasi-error in the dual problem will be non-increasing, and vice-versa. As such, the procedures SOLVE and ESTIMATE are performed for each of the primal and dual problems. The marked set is taken to be the union of marked sets from the primal and dual problems, each chosen to satisfy the Dörfler property. This method produces a sequence of refinements for which both the error in the primal and dual problems contract at each step.

Procedure SOLVE. The contraction result supposes the exact Galerkin solution is found on each mesh refinement. In practice a linear-time iterative method is employed so that the Galerkin solution is found up to a given tolerance.

Procedure ESTIMATE. The estimation of the error on each element is determined by a standard residual-based estimator. The residuals over element interiors and jump-residuals over the boundaries are based on the *local strong forms* of the elliptic operator and its adjoint as follows.

$$\mathcal{L}(v) = \nabla \cdot (A\nabla v) - b \cdot \nabla v - cv; \quad \mathcal{L}^*(v) = \nabla \cdot (A\nabla v) + b \cdot \nabla v - cv. \tag{2.24}$$

The *residuals* for the primal and dual problems using the sign convention in [4] are:

$$R(v) := f + \mathcal{L}(v); \quad R^*(v) := g + \mathcal{L}^*(v),\ v \in \mathbb{V}_{\mathcal{T}}. \tag{2.25}$$

While the primal and dual solutions u and z of (1.3) and (2.1) respectively satisfy

$$f(z) = a(u, z) = a^*(z, u) = g(u)$$

the residuals for the primal and dual problems are in general different. The *jump residual* for the primal and dual problems is

$$J_T(v) := [\![[A\nabla v] \cdot n]\!]_{\partial T} \tag{2.26}$$

where *jump operator* $[\![\,\cdot\,]\!]$ is given by

$$[\![\phi]\!]_{\partial T} := \lim_{t\to 0} \phi(x+tn) - \phi(x-tn) \tag{2.27}$$

and n is taken to be the appropriate outward normal defined piecewise on ∂T. On boundary edges σ_b we have

$$[\![[A\nabla v]\cdot n]\!]_{\sigma_b} \equiv 0$$

so that $[\![[A\nabla v]\cdot n]\!]_{\partial T} = [\![[A\nabla v]\cdot n]\!]_{\partial T\cap\Omega}$. For clarity, we will also employ the notation

$$R_T(v) := R(v)\big|_T,\ v \in \mathbb{V}_{\mathcal{T}},$$

and similarly for the other strong form operators. The error indicator is given as

$$\eta^p_{\mathcal{T}}(v,T) := h_T^p\|R(v)\|^p_{L_2(T)} + h_T^{p/2}\|J_T(v)\|^p_{L_2(\partial T)}, \quad v\in\mathbb{V}_{\mathcal{T}}. \tag{2.28}$$

The dual error-indicator is then given by

$$\zeta^p_{\mathcal{T}}(w,T) := h_T^p\|R^*(w)\|^p_{L_2(T)} + h_T^{p/2}\|J_T(w)\|^p_{L_2(\partial T)}, \quad w\in\mathbb{V}_{\mathcal{T}}. \tag{2.29}$$

The error estimators are given by the l_p sum of error indicators over elements in the space where $p=1$ or 2.

$$\eta^p_{\mathcal{T}}(v) := \sum_{T\in\mathcal{T}} \eta^p_{\mathcal{T}}(v,T), \quad v\in\mathbb{V}_{\mathcal{T}}. \tag{2.30}$$

The dual energy estimator is:

$$\zeta^p_{\mathcal{T}}(w) := \sum_{T\in\mathcal{T}} \zeta^p_{\mathcal{T}}(w), \quad w\in\mathbb{V}_{\mathcal{T}}. \tag{2.31}$$

The contraction results for the quasi-error presented below will be shown to hold for $p=1,2$ where the error estimator and oscillation are defined in terms of the l_p norm. While complexity results are shown only for $p=2$, the contraction results for $p=1$ are useful for nonlinear problems; see [8].

For analyzing oscillation, for $v\in\mathbb{V}_{\mathcal{T}}$ let Π^2_m the orthogonal projector defined by the best L_2 approximation in $\mathbb{P}_m$ over mesh $\mathcal{T}$ and $P^2_m = I-\Pi^2_m$. Define now the oscillation on the elements $T\in\mathcal{T}$ for the primal problem by

$$\mathrm{osc}_{\mathcal{T}}(v,T) := h_T\|P^2_{2n-2}R(v)\|_{L_2(T)} \tag{2.32}$$

and analogously for the dual problem. For subsets $\omega\subseteq\mathcal{T}$ set

$$\mathrm{osc}^p_{\mathcal{T}}(v,\omega) := \sum_{T\in\omega}\mathrm{osc}^p_{\mathcal{T}}(v,T). \tag{2.33}$$

The data estimator and data oscillation, identical for both the primal and dual problems, are given by

$$\eta^p_{\mathcal{T}}(D,T) := h_T^p\left(\|\mathrm{div}A\|^p_{L_\infty(T)} + h_T^{-p}\|A\|^p_{L_\infty(\omega_T)} + \|c\|^p_{L_\infty(T)} + \|b\|^p_{L_\infty(T)}\right), \tag{2.34}$$

$$\begin{aligned}\mathrm{osc}^p_{\mathcal{T}}(D,T) := h_T^p\Big(&\|P^\infty_{n-1}\mathrm{div}A\|^p_{L_\infty(T)} + h_T^{-p}\|P^\infty_n A\|^p_{L_\infty(T)}\\ &+h_T^p\|P^\infty_{n-2}c\|^p_{L_\infty(T)} + \|P^\infty_{2n-2}c\|^p_{L_\infty(T)} + \|P^\infty_{n-1}b\|^p_{L_\infty(T)}\Big).\end{aligned} \tag{2.35}$$

The data estimator and oscillation over the mesh $\mathcal{T}$ or a subset $\omega\subset\mathcal{T}$ are given by the maximum data estimator (oscillation) over elements in the mesh or subset: For $\omega\subseteq\mathcal{T}$

$$\eta_{\mathcal{T}}(D,\omega) = \max_{T\in\omega}\eta_{\mathcal{T}}(D,T) \text{ and } \mathrm{osc}_{\mathcal{T}}(D,\omega) = \max_{T\in\omega}\mathrm{osc}_{\mathcal{T}}(D,T).$$

The data estimator and data oscillation on the initial mesh

$$\eta_0 := \eta_{\mathcal{T}_0}(D,\mathcal{T}_0), \ \text{ and } \ \mathrm{osc}_0 := \mathrm{osc}_{\mathcal{T}_0}(D,\mathcal{T}_0).$$

As the grid is refined, the data estimator and data oscillation terms satisfy the monotonicity property [4] for refinements $\mathcal{T}_2 \geq \mathcal{T}_1$

$$\eta_2(D, \mathcal{T}_2) \leq \eta_1(D, \mathcal{T}_1) \text{ and } \mathrm{osc}_2(D, \mathcal{T}_2) \leq \mathrm{osc}_1(D, \mathcal{T}_1). \tag{2.36}$$

Procedure MARK. The Dörfler marking strategy for the goal-oriented problem is based on the following steps as in [10]:

1) Given $\theta \in (0, 1)$, mark sets for each of the primal and dual problems:
   - Mark a set $\mathcal{M}_p \subset \mathcal{T}_k$ such that,
   $$\sum_{T \in \mathcal{M}_p} \eta_k^2(u_k, T) \geq \theta^2 \eta_k^2(u_k, \mathcal{T}_k) \tag{2.37}$$
   - Mark a set $\mathcal{M}_d \subset \mathcal{T}_k$ such that,
   $$\sum_{T \in \mathcal{M}_d} \zeta_k^2(z_k, T) \geq \theta^2 \zeta_k^2(z_k, \mathcal{T}_k) \tag{2.38}$$
2) Let $\mathcal{M} = \mathcal{M}_p \cup \mathcal{M}_d$ the union of sets found for the primal and dual problems respectively.

The set $\mathcal{M}$ differs from that in [10], where the set of lesser cardinality between $\mathcal{M}_p$ and $\mathcal{M}_d$ is used. In the case of the nonsymmetric problem the error reduced at each iteration is the quasi-error rather than the energy error as in the symmetric problem [10]. This error for each problem is guaranteed to contract based on the refinement satisfying the Dörfler property. As such, refining the mesh with respect to one problem does not guarantee the quasi-error in the other problem is nonincreasing. Sets $\mathcal{M}_p$ and $\mathcal{M}_d$ with optimal cardinality (up to a factor of 2) can be chosen in linear time by binning the elements rather than performing a full sort [10].

Procedure REFINE. The refinement (including the completion) is performed according to newest vertex bisection [2]. The complexity and other properties of this procedure are now well-understood, and will simply be exploited here.

3. CONTRACTION AND CONVERGENCE THEOREMS

The key elements of the main contraction argument constructed below are quasi-orthogonality 3.1, error estimator as upper-bound on energy-norm error 3.2 and estimator reduction 3.4. Estimator-reduction is shown via the local-perturbation estimate 3.3. The local perturbation of the oscillation is presented here and used in §4. Mesh refinements $\mathcal{T}_1$ and $\mathcal{T}_2$ (respectively $\mathcal{T}_j$) are assumed conforming, and u_j is assumed the Galerkin solution on refinement $\mathcal{T}_j$. The following results hold for both the primal and dual problems which differ by the sign of the convection term; therefore, they are established here only for the primal problem.

3.1. **Quasi-orthogonality.**

Orthogonality in the energy-norm $|||u - u_2|||^2 = |||u - u_1|||^2 - |||u_2 - u_1|||^2$ does not generally hold in the nonsymmetric problem. We use the weaker quasi-orthogonality result to establish contraction of AFEM (GOAFEM). The following is a variation on Lemma 2.1 in [9] (see also [8]).

Lemma 3.1 (Quasi-orthogonality)**.** *Let the problem data satisfy Assumption 2.1 and the mesh satisfy conditions (1) and (2) of Assumption 2.2. Let* $\mathcal{T}_1, \mathcal{T}_2 \in \mathbb{T}$ *with* $\mathcal{T}_2 \geq \mathcal{T}_1$*. Let* $u_k \in \mathbb{V}_k$ *the solution to* (2.22)*,* $k = 1, 2$*. There exists a constant* $C_* > 0$ *depending on the problem data* D *and initial mesh* $\mathcal{T}_0$*, and a number* $0 < s \leq 1$ *dictated only by the angles* *of $\partial\Omega$, such that if the meshsize h_0 of the initial mesh satisfies $\bar{\Lambda} \coloneqq C_* h_0^s \|b\|_{L_\infty} \mu_0^{-1/2} < 1$, then*

$$|\!|\!| u - u_2 |\!|\!|^2 \leq \Lambda |\!|\!| u - u_1 |\!|\!|^2 - |\!|\!| u_2 - u_1 |\!|\!|^2, \tag{3.1}$$

where

$$\Lambda \coloneqq (1 - C_* h_0^s \|b\|_{L_\infty} \mu_0^{-1/2})^{-1}.$$

Equality holds (usual orthogonality) when $b = 0$ in Ω, in which case the problem is symmetric.

Proof. The proof follows close that of Lemma 2.1 in [9]. Let

$$e_2 \coloneqq u - u_2, \quad e_1 \coloneqq u - u_1, \quad \text{and } \varepsilon_1 \coloneqq u_2 - u_1.$$

By Galerkin orthogonality

$$|\!|\!| e_1 |\!|\!|^2 = a(e_1, e_1) = |\!|\!| e_2 |\!|\!|^2 + |\!|\!| \varepsilon_1 |\!|\!|^2 + a(\varepsilon_1, e_2). \tag{3.2}$$

Rearranging and applying the divergence-free condition on the convection term

$$|\!|\!| e_2 |\!|\!|^2 = |\!|\!| e_1 |\!|\!|^2 - |\!|\!| \varepsilon_1 |\!|\!|^2 - 2\langle b \cdot \nabla \varepsilon_1, e_2 \rangle.$$

Applying Hölder's inequality and coercivity (2.9) $|\varepsilon_1|_{H^1} \leq \mu_0^{-1/2} |\!|\!| \varepsilon_1 |\!|\!|$ followed by Young's inequality with constant δ to be determined,

$$-2\langle b \cdot \nabla \varepsilon_1, e_2 \rangle \leq \delta \|e_2\|_{L_2}^2 + \frac{\|b\|_{L_\infty}^2}{\delta \mu_0} |\!|\!| \varepsilon_1 |\!|\!|^2. \tag{3.3}$$

By a duality argument for some $C_* > 0$ assuming $u \in H^{1+s}(\Omega)$ for some $0 < s \leq 1$ depending on the angles of $\partial\Omega$

$$\|e_2\|_{L_2} \leq C_* h_0^s |\!|\!| e_2 |\!|\!|. \tag{3.4}$$

The details of this argument as described in the appendix §6 may also be found in [1] and [5]. Applying (3.4) and (3.3) to (3.2),

$$(1 - \delta C_*^2 h_0^{2s}) |\!|\!| u - u_2 |\!|\!|^2 \leq |\!|\!| u - u_1 |\!|\!|^2 - \left(1 - \frac{\|b\|_{L_\infty}^2}{\delta \mu_0}\right) |\!|\!| u_1 - u_2 |\!|\!|^2. \tag{3.5}$$

Choose δ to equate coefficients

$$\delta C_*^2 h_0^{2s} = \frac{\|b\|_{L_\infty}^2}{\delta \mu_0} \implies \delta = \frac{\|b\|_{L_\infty}}{C_* h_0^s \sqrt{\mu_0}},$$

then

$$|\!|\!| u - u_2 |\!|\!|^2 \leq \left(1 - \|b\|_{L_\infty} C_* h_0^s \mu_0^{-1/2}\right)^{-1} |\!|\!| u - u_1 |\!|\!|^2 - |\!|\!| u_1 - u_2 |\!|\!|^2.$$

Assuming the initial mesh as characterized by h_0 satisfies

$$\bar{\Lambda} = \|b\|_{L_\infty} C_* h_0^s \mu_0^{-1/2} < 1, \tag{3.6}$$

the quasi-orthogonality result holds. □

Note that by (3.2) we also have

$$|\!|\!| \varepsilon_1 |\!|\!|^2 = |\!|\!| e_1 |\!|\!|^2 - |\!|\!| e_2 |\!|\!|^2 - 2\langle b \cdot \nabla e_2, \varepsilon_1 \rangle. \tag{3.7}$$

Similarly to (3.3)

$$-2\langle b \cdot \nabla e_2, \varepsilon_1 \rangle \geq -2|\langle b \cdot \nabla e_2, \varepsilon_1 \rangle| \geq -\delta \|\varepsilon_1\|_{L_2}^2 - \frac{\|b\|_{L_\infty}^2}{\delta \mu_0} |\!|\!| e_2 |\!|\!|^2, \tag{3.8}$$

which under the same assumptions yields the estimate

$$|||u_2 - u_1|||^2 \geq (1+\bar{\Lambda})^{-1} |||u - u_1|||^2 - |||u - u_2|||^2, \tag{3.9}$$

where $\bar{\Lambda} < 1 \implies (1+\bar{\Lambda})^{-1} > 1/2$.

3.2. **Error estimator as global upper-bound.** We now recall the property that the error estimator is a global upper bound on the error. The proof is fairly standard; see e.g. [10] (Proposition 4.1), [9] (3.6), and [8].

Lemma 3.2 (Error estimator as global upper-bound)**.** *Let the problem data satisfy Assumption 2.1 and the mesh satisfy conditions (1) and (2) of Assumption 2.2. Let* $\mathcal{T}_1, \mathcal{T}_2 \in \mathbb{T}$ *with* $\mathcal{T}_2 \geq \mathcal{T}_1$*. Let* $u_k \in \mathbb{V}_k$ *the solution to* (2.22), $k = 1, 2$ *and* u *the solution to* (1.3)*. Let*

$$G = G(\mathcal{T}_2, \mathcal{T}_1) \coloneqq \{T \subset \mathcal{T}_1 \mid T \cap \tilde{T} \neq \emptyset \text{ for some } \tilde{T} \in \mathcal{T}_1, \tilde{T} \notin \mathcal{T}_2\}.$$

Then for global constant C_1 *depending on the problem data* D *and initial mesh* $\mathcal{T}_0$

$$|||u_2 - u_1||| \leq C_1 \eta_1(u_1, G) \tag{3.10}$$

and in particular

$$|||u - u_1||| \leq C_1 \eta_1(u_1, \mathcal{T}_1). \tag{3.11}$$

3.3. **Local perturbation.** The local perturbation property established in [4], analogous to the local Lipshitz property in [8], is a key step in establishing the contraction result. This is a minor variation on Proposition 3.3 in [4] which deals with a symmetric problem. Here, we include a convection term in the estimate. In particular, (3.12) shows that the difference in the error indicators over an element T between two functions in a given finite element space may be bounded by a fixed factor of the native norm over the patch ω_T of the difference in functions. In contrast with the analogous result in [4] the estimate (3.13) involves a fixed factor of the native norm over an individual element rather than a patch as by the continuity of A the oscillation term does not involve the jump residual.

We include the proof of (3.12) for completeness. The proof of (3.13) may be found in [4] with the final result inferred by the absence of the jump residual in the oscillation term.

Lemma 3.3 (Local perturbation)**.** *Let the problem data satisfy Assumption 2.1 and the mesh satisfy condition (1) of Assumption 2.2. Let* $\mathcal{T} \in \mathbb{T}$*. For all* $T \in \mathcal{T}$ *and for any* $v, w \in \mathbb{V}_{\mathcal{T}}$

$$\eta_{\mathcal{T}}(v, T) \leq \eta_{\mathcal{T}}(w, T) + \bar{\Lambda}_1 \eta_{\mathcal{T}}(D, T) \|v - w\|_{H^1(\omega_T)} \tag{3.12}$$

$$\text{osc}_{\mathcal{T}}(v, T) \leq \text{osc}_{\mathcal{T}}(w, T) + \bar{\Lambda}_2 \text{osc}_{\mathcal{T}}(D, T) \|v - w\|_{H^1(T)} \tag{3.13}$$

where recalling (2.21) ω_T *is the union of* T *with elements in* $\mathcal{T}$ *sharing a true-hyperface with* T*. The constants* $\bar{\Lambda}_1, \bar{\Lambda}_2 > 0$ *depend on the initial mesh* $\mathcal{T}_0$*, the dimension* d *and the polynomial degree* n*.*

Proof of (3.12)*.* From (2.28)

$$\eta^p_{\mathcal{T}}(v, T) \coloneqq h^p_T \|R(v)\|^p_{L_2(T)} + h^{p/2}_T \|J_T(v)\|^p_{L_2(\partial T)}, \quad v \in \mathbb{V}_{\mathcal{T}}. \tag{3.14}$$

Denote $\eta_{\mathcal{T}}(v, T)$ by $\eta(v, T)$. Set $e = v - w$. By linearity

$$R(v) = R(w + e) = f + \mathcal{L}(w + e) = f + \mathcal{L}(w) + \mathcal{L}(e) = R(w) + \mathcal{L}(e)$$

and

$$J(v) = J(w + e) = J(w) + J(e).$$

For $p = 1$ by the triangle inequality

$$\eta(v,T) = h_T\|R(w) + \mathcal{L}(e)\|_{L_2(T)} + h_T^{1/2}\|J(w) + J(e)\|_{L_2(\partial T)}$$
$$\leq \eta(w,T) + h_T\|\mathcal{L}(e)\|_{L_2(T)} + h_T^{1/2}\|J(e)\|_{L_2(\partial T)}.$$

For $p = 2$ using the generalized triangle-inequality

$$\sqrt{(a+b)^2 + (c+d)^2} \leq \sqrt{a^2+c^2} + b + d, \quad \text{for } a,b,c,d > 0 \tag{3.15}$$

we have

$$\eta(v,T) = \left(h_T^2\|R(w) + \mathcal{L}(e)\|^2_{L_2(T)} + h_T\|J(w) + J(e)\|^2_{L_2(\partial T)}\right)^{1/2}$$
$$\leq \eta(w,T) + h_T\|\mathcal{L}(e)\|_{L_2(T)} + h_T^{1/2}\|J(e)\|_{L_2(\partial T)}.$$

Consider the second term on the RHS $h_T\|\mathcal{L}(e)\|_{L_2(T)}$. By definition (2.24) of $\mathcal{L}(\,\cdot\,)$, the product rule applied to the diffusion term and the triangle-inequality

$$\|\mathcal{L}(e)\|_{L_2(T)} \leq \|\mathrm{div}A\cdot\nabla e\|_{L_2(T)} + \|A : D^2e\|_{L_2(T)} + \|ce\|_{L_2(T)} + \|b\cdot\nabla e\|_{L_2(T)}$$

where D^2e is the Hessian of e. Consider each term. The first diffusion term

$$\|\mathrm{div}A\cdot\nabla e\|_{L_2(T)} \leq \|\mathrm{div}A\|_{L_\infty(T)}\|\nabla e\|_{L_2(T)} \tag{3.16}$$

by the inequality

$$\|v\cdot z\|_{L_2(T)} \leq \|v\|_{L_\infty(T)}\|z\|_{L_2(T)}, \quad v \in L_\infty(T),\ z \in L_2(T). \tag{3.17}$$

Applying (3.17) and inverse-estimate [3] to the second diffusion term

$$\|A : D^2e\|_{L_2(T)} \leq \|A\|_{L_\infty(T)}\|D^2e\|_{L_2(T)}$$
$$\leq C_I h_T^{-1}\|A\|_{L_\infty(T)}\|\nabla e\|_{L_2(T)}. \tag{3.18}$$

For the reaction term

$$\|ce\|_{L_2(T)} \leq \|c\|_{L_\infty(T)}\|e\|_{L_2(T)}. \tag{3.19}$$

For the convection term applying (3.17)

$$\|b\cdot\nabla e\|_{L_2(T)} \leq \|b\|_{L_\infty(T)}\|\nabla e\|_{L_2(T)}. \tag{3.20}$$

Consider the the jump-residual term $\|J(e)\|_{L_2(\partial T)}$. For each interior true-hyperface $\sigma = T\cap T',\ T,T' \in \mathcal{T}$ by (2.27)

$$J(e)\big|_\sigma := \lim_{t\to 0^+}(A\nabla e)(x + tn_\sigma) - \lim_{t\to 0^-}(A\nabla e)(x - tn_\sigma)$$
$$= n_\sigma\cdot(A\nabla e)\big|_T - n_\sigma\cdot(A\nabla e)\big|_{T'} \tag{3.21}$$

where $(A\nabla e)\big|_T$ is understood to refer to the product of the limiting value of $A\nabla e$ as the element boundary is approached from the interior of T. By the triangle-inequality

$$\|J(e)\|_{L_2(\sigma)} \leq \|n_\sigma\cdot(A\nabla e)\big|_T\|_{L_2(\sigma)} + \|n_\sigma\cdot(A\nabla e)\big|_{T'}\|_{L_2(\sigma)}.$$

By bounds for the inner-product with a unit normal and a matrix-vector product

$$\|\phi\cdot n\|_{L_2(\sigma)} \leq \|\phi\|_{L_2(\sigma)}, \quad \phi \in L_2(\sigma), \tag{3.22}$$
$$\|M\phi\|_{L_2(T)} \leq \|M\|_{L_\infty(T)}\|\phi\|_{L_2(T)}, \quad M \in L_\infty(T),\ \phi \in L_2(T) \tag{3.23}$$

obtain

$$\|n_\sigma\cdot(A\nabla e)\big|_T\|_{L_2(\sigma)} \leq \|(A\nabla e)\big|_T\|_{L_2(\sigma)} \leq \|A\big|_T\|_{L_\infty(\sigma)}\|\nabla e\big|_T\|_{L_2(\sigma)}. \tag{3.24}$$

Applying the trace theorem and an inverse inequality to $\|\nabla e\big|_T\|_{L_2(\sigma)}$ via the inequality

$$\|\phi\|_{L_2(\sigma)} \leq Ch_T^{-1/2}\|\phi\|_{L_2(T)}, \quad \phi \in L_2(T) \tag{3.25}$$

we have

$$\|\nabla e\big|_T\|_{L_2(\sigma)} \leq C_T(\bar{\gamma})^{d-1} h_T^{-1/2} \|\nabla e\|_{L_2(T)}. \tag{3.26}$$

By the Lipschitz property of A

$$\|A\big|_T\|_{L_\infty(\sigma)} = \|A\|_{L_\infty(\sigma)} \leq \|A\|_{L_\infty(T)}. \tag{3.27}$$

By (3.24), (3.26), (3.27) and comparability of mesh diameters (2.15)

$$\|J(e)\|_{L_2(\sigma)} \leq 2C_T(\bar{\gamma})^{d-1}\gamma_N^{1/2} h_T^{-1/2} \|A\|_{L_\infty(\omega_T)} \|\nabla e\|_{L_2(\omega_T)}.$$

Element T has at most $d+1$ interior true-hyperfaces yielding

$$\begin{aligned} \|J(e)\|_{L_2(\partial T)} &\leq 2(d+1)\, C_T(\bar{\gamma})^{d-1}\gamma_N^{1/2} h_T^{-1/2} \|A\|_{L_\infty(\omega_T)} \|\nabla e\|_{L_2(\omega_T)} \\ &= C_J h_T^{-1/2} \|A\|_{L_\infty(\omega_T)} \|\nabla e\|_{L_2(\omega_T)}. \end{aligned}$$

Putting together the terms from $\mathcal{L}$ and from the jump residual,

$$\begin{aligned} \eta(v,T) \leq\ & \eta(w,T) + h_T \left(\|\mathrm{div} A\|_{L_\infty(T)} + C_I h_T^{-1} \|A\|_{L_\infty(T)} \right. \\ & \left. + \|c\|_{L_\infty(T)} + \|b\|_{L_\infty(\omega)} \right) \|e\|_{H^1(T)} + h_T^{1/2} C_J h_T^{-1/2} \|A\|_{L_\infty(\omega_T)} \|e\|_{H^1(\omega_T)} \\ \leq\ & \eta(w,T) + C_{TOT'}\, \eta_T(D,T) \|v-w\|_{H^1(\omega_T)} \end{aligned}$$

where $C_{TOT'}$ differs by a factor of 2 for $p = 1, 2$. □

3.4. **Estimator reduction.** We now establish one of the three key results we need, namely estimator reduction. This result is a minor variation of [4] Corollary 2.4 and is stated here for completeness.

Theorem 3.4 (Estimator reduction)**.** *Let the problem data satisfy Assumption 2.1 and the mesh satisfy conditions (1) and (2) of Assumption 2.2. Let $\mathcal{T}_1 \in \mathbb{T}$, $\mathcal{M} \subset \mathcal{T}_1$ and $\mathcal{T}_2 =$ REFINE$(\mathcal{T}_1, \mathcal{M})$. For $p = 1$ let*

$$\Lambda_1 := (d+2)^2 \bar{\Lambda}_1^2 m_{\mathcal{E}}^{-2} \quad \textit{and} \ \ \lambda := (1 - 2^{-1/2d})^2 > 0$$

and for $p = 2$ let

$$\Lambda_1 := (d+2) \bar{\Lambda}_1^2 m_{\mathcal{E}}^{-2} \quad \textit{and} \ \ \lambda := 1 - 2^{-1/d} > 0$$

with $\bar{\Lambda}_1$ from 3.3 (Local Perturbation). Then for any $v_1 \in \mathbb{V}_1$ and $v_2 \in \mathbb{V}_2$ and $\delta > 0$

$$\eta_2^2(v_2, \mathcal{T}_2) \leq (1+\delta)\left\{ \eta_1^2(v_1, \mathcal{T}_1) - \lambda \eta_1^2(v_1, \mathcal{M}) \right\} + (1+\delta^{-1}) \Lambda_1 \eta_0^2 |\!|\!| v_2 - v_1 |\!|\!|^2. \tag{3.28}$$

Proof. The proofs for $p = 1$ and $p = 2$ are similar. For $p = 1$ it is necessary to sum over elements before squaring and for $p = 2$ square first then sum over elements.

Proof for the case $p = 1$. By the local Lipschitz property (3.12)

$$\eta_2(v_2, T) \leq \eta_2(v_1, T) + \bar{\Lambda}_1 \eta_2(D,T) \|v_2 - v_1\|_{H^1(\omega_T)}. \tag{3.29}$$

Summing over all elements $T \in \mathcal{T}_2$, the sum of norms over ω_T covers each element at most $(d+2)$ times as each patch ω_T is the union of element T and the (up to) $d+1$ elements sharing a true-hyperface with T. Then by the coercivity (2.12) over Ω

$$\eta_2(v_2, \mathcal{T}_2) \leq \eta_2(v_1, \mathcal{T}_2) + (d+2)\bar{\Lambda}_1 m_{\mathcal{E}}{}^{-1} \eta_2^2(D, \mathcal{T}_2) |\!|\!| v_2 - v_1 |\!|\!|. \tag{3.30}$$

Squaring (3.30) and applying Young's inequality with constant δ to the cross-term,

$$\begin{aligned} \eta_2^2(v_2, \mathcal{T}_2) &\leq (1+\delta)\eta_2^2(v_1, \mathcal{T}_2) + (1+\delta^{-1})(d+2)^2 \bar{\Lambda}_1^2 m_{\mathcal{E}}^{-2} \eta_2^2(D, \mathcal{T}_2) |\!|\!| v_2 - v_1 |\!|\!|^2 \\ &= (1+\delta)\eta_2^2(v_1, \mathcal{T}_2) + (1+\delta^{-1}) \Lambda_1 \eta_2^2(D, \mathcal{T}_2) |\!|\!| v_2 - v_1 |\!|\!|^2. \end{aligned} \tag{3.31}$$

For an element $T \in \mathcal{M}$ marked for refinement, let $\mathcal{T}_{2,T} \coloneqq \{T' \in \mathcal{T}_2 \bigm| T' \subset T\}$. As $v_1 \in \mathbb{V}_1$ has no discontinuities across element boundaries in $\mathcal{T}_{2,T}$, we have $J(v_1) = 0$ on true hyperfaces in the interior of $\mathcal{T}_{2,T}$.

Recall the element diameter $h_T = |T|^{1/d}$. For an element T marked for refinement, T' must be a proper subset of T, in particular a product of at least one bisection so that

$$|T'| \le \frac{1}{2}|T| \leftrightarrow |T'|^{1/d} \le \frac{1}{2^{1/d}}|T|^{1/d} \leftrightarrow h_{T'} \le \frac{1}{2^{1/d}}h_T. \tag{3.32}$$

Then

$$\begin{aligned}\sum_{T'\in\mathcal{T}_{2,T}} \eta_2(v_1,T') &\le \sum_{T'\in\mathcal{T}_{2,T}} h_{T'}\|R(v_1)\|_{L_2(T')} + \sum_{T'\in\mathcal{T}_{2,T}} h_{T'}^{1/2}\|J(v)\|_{L_2(\partial T'\cap\partial T)} \\ &\le 2^{-1/d}h_T \sum_{T'\in\mathcal{T}_{2,T}} \left(\|R(v_1)\|_{L_2(T')}\right) + 2^{-1/2d}h_T^{1/2}\|J(v)\|_{L_2(\partial T)} \\ &\le 2^{-1/2d}\left(h_T\|R(v_1)\|_{L_2(T)} + h_T^{1/2}\|J(v)\|_{L_2(\partial T)}\right) \\ &= 2^{-1/2d}\eta_1(v_1,T).\end{aligned} \tag{3.33}$$

For an element $T \notin \mathcal{M}$, that is $T' = T$ the indicator is reproduced

$$\eta_2(v_1,T') = \eta_1(v_1,T). \tag{3.34}$$

Sum over all $T \in \mathcal{T}_2$ by estimates (3.33), (3.34) writing the sum of indicators over the $\mathcal{T}_1 \setminus \mathcal{M}$ as the total estimator less the indicators over the refinement set $\mathcal{M}$. Let the refined set $\mathcal{R} \coloneqq \{T \in \mathcal{T}_2 \bigm| T' \subset \tilde{T} \text{ for some } \tilde{T} \in \mathcal{M}\}$ then

$$\begin{aligned}\eta_2(v_1,\mathcal{T}_2) &= \sum_{T\in\mathcal{T}_2} \eta_2(v_1,T) \\ &= \sum_{T\in\mathcal{T}_2\setminus\mathcal{R}} \eta_2(v_1,T) + \sum_{T\in\mathcal{R}} \eta_2(v_1,T) \\ &\le \eta_1(v_1,\mathcal{T}_1) - \eta_1(v_1,\mathcal{M}) + 2^{-1/2d}\eta_1(v_1,\mathcal{M}) \\ &= \eta_1(v_1,\mathcal{T}_1) - \lambda_1\,\eta_1(v_1,\mathcal{M})\end{aligned} \tag{3.35}$$

where $\lambda_1 = 1 - 2^{-1/2d} < 1$. Squaring (3.35)

$$\begin{aligned}\eta_2^2(v_1,\mathcal{T}_2) &\le \eta_1^2(v_1,\mathcal{T}_1) + \lambda_1^2\,\eta_1^2(v_1,\mathcal{M}) - 2\lambda_1^2\,\eta_1^2(v_1,\mathcal{M}) \\ &= \eta_1^2(v_1,\mathcal{T}_1) - \lambda\,\eta_1^2(v_1,\mathcal{M})\end{aligned} \tag{3.36}$$

where $\lambda = \lambda_1^2 = (1 - 2^{-1/2d})^2$. Applying (3.36) to (3.31) and applying monotonicity of the data-estimator

$$\begin{aligned}\eta_2^2(v_2,\mathcal{T}_2) \le{}& (1+\delta)\left(\eta_1^2(v_1,\mathcal{T}_1) - \lambda\,\eta_1^2(v_1,\mathcal{M})\right) \\ &+ (1+\delta^{-1})\Lambda_1^2\eta_0^2(D,\mathcal{T}_0)|\!|\!|v_2 - v_1|\!|\!|^2.\end{aligned}$$

The proof for the case $p = 2$ is similar and may be found in [4]. □

3.5. **Contraction of AFEM.** We now establish the main contraction results. The contraction result 3.5 is a modification of [4] Theorem 4.1. Here we use quasi-orthogonality to establish contraction of each of the nonsymmetric problems (1.3) and (2.1).

Theorem 3.5 (GOAFEM contraction)**.** *Let the problem data satisfy Assumption 2.1 and the mesh satisfy Assumption 2.2. Let u the solution to* (1.3)*. Let $\theta \in (0,1]$, and let $\{\mathcal{T}_k, \mathbb{V}_k, u_k\}_{k\ge 0}$ be the sequence of meshes, finite element spaces and discrete solutions* *produced by GOAFEM. Then there exist constants* $\gamma > 0$ *and* $0 < \alpha < 1$*, depending on the initial mesh* $\mathcal{T}_0$ *and marking parameter* θ *such that*

$$\|u - u_{k+1}\|^2 + \gamma\eta_{k+1}^2 \le \alpha^2 \left(\|u - u_k\|^2 + \gamma\eta_k^2\right). \tag{3.37}$$

The analogous result holds for the dual problem with $\{\mathcal{T}_k, \mathbb{V}_k, z_k\}_{k\ge 0}$ *the sequence of meshes, finite element spaces and discrete solutions produced by GOAFEM.*

Proof. Denote

$$e_k = u - u_k, \quad e_{k+1} = u - u_{k+1} \quad \text{and} \quad \varepsilon_k = u_{k+1} - u_k.$$

Let

$$\eta_k = \eta_k(u_k, \mathcal{T}_k), \quad \eta_k(\mathcal{M}_k) = \eta_k(u_k, \mathcal{M}_k) \quad \text{and} \quad \eta_{k+1} = \eta_{k+1}(u_{k+1}, \mathcal{T}_{k+1}).$$

By the result of Estimator Reduction 3.4, for any $\delta > 0$

$$\eta_{k+1}^2 \le (1+\delta)\left\{\eta_k^2 - \lambda\eta_k^2(\mathcal{M}_k)\right\} + (1+\delta^{-1})\Lambda_1\eta_0^2\|\varepsilon_k\|^2.$$

Multiplying this inequality by positive constant γ (to be determined) and adding the quasi-orthogonality estimate $\|e_{k+1}\|^2 \le \Lambda\|e_k\|^2 - \|\varepsilon_k\|^2$ obtain

$$\begin{aligned}\|e_{k+1}\|^2 + \gamma\eta_{k+1}^2 \le \Lambda\|e_k\|^2 - \|\varepsilon_k\|^2 + \gamma(1+\delta)\left\{\eta_k^2 - \lambda\eta_k^2(\mathcal{M}_k)\right\}\\ + \gamma(1+\delta^{-1})\Lambda_1\eta_0^2\|\varepsilon_k\|^2.\end{aligned} \tag{3.38}$$

Choose γ to eliminate $\|\varepsilon_k\|$ the error between consecutive estimates by setting

$$\gamma(1+\delta^{-1})\Lambda_1\eta_0^2 = 1 \iff \gamma = \frac{1}{(1+1/\delta)\Lambda_1\eta_0^2} \iff \gamma(1+\delta) = \frac{\delta}{\Lambda_1\eta_0^2}. \tag{3.39}$$

Applying (3.39) to (3.38) obtain

$$\|e_{k+1}\|^2 + \gamma\eta_{k+1}^2 \le \Lambda\|e_k\|^2 + \gamma(1+\delta)\eta_k^2 - \gamma(1+\delta)\lambda\eta_k^2(\mathcal{M}_k). \tag{3.40}$$

By the Dörfler marking strategy $\eta_k^2(\mathcal{M}_k) \ge \theta^2\eta_k^2$ so that

$$\|e_{k+1}\|^2 + \gamma\eta_{k+1}^2 \le \Lambda\|e_k\|^2 + \gamma(1+\delta)\eta_k^2 - \gamma(1+\delta)\lambda\theta^2\eta_k^2. \tag{3.41}$$

Split the last term by factors of β and $(1-\beta)$ for any $\beta \in (0,1)$ to arrive at

$$\begin{aligned}\|e_{k+1}\|^2 + \gamma\eta_{k+1}^2 \le \Lambda\|e_k\|^2 + \gamma(1+\delta)\eta_k^2 - \beta\gamma(1+\delta)\lambda\theta^2\eta_k^2\\ - (1-\beta)\gamma(1+\delta)\lambda\theta^2\eta_k^2.\end{aligned} \tag{3.42}$$

Applying the upper-bound estimate (3.11) $\|e_k\|^2 \le C_1\eta_k^2$ to the term multiplied by β then by (3.39)

$$\begin{aligned}\|e_{k+1}\|^2 + \gamma\eta_{k+1}^2 &\le \Lambda\|e_k\|^2 - \frac{\beta\gamma(1+\delta)\lambda\theta^2}{C_1}\|e_k\|^2 + \gamma(1+\delta)\eta_k^2\\ &\qquad - (1-\beta)\gamma(1+\delta)\lambda\theta^2\eta_k^2 &(3.43)\\ &= \Lambda\|e_k\|^2 - \beta\frac{\delta\lambda\theta^2}{C_1\Lambda_1\eta_0^2}\|e_k\|^2 + \gamma(1+\delta)\eta_k^2\\ &\qquad - (1-\beta)\gamma(1+\delta)\lambda\theta^2\eta_k^2 &(3.44)\\ &= \left(\Lambda - \beta\frac{\delta\lambda\theta^2}{C_1\Lambda_1\eta_0^2}\right)\|e_k\|^2 + \gamma(1+\delta)\left(1-(1-\beta)\lambda\theta^2\right)\eta_k^2 &(3.45)\\ &= \alpha_1^2(\delta,\beta)\|e_k\|^2 + \gamma\alpha_2^2(\delta,\beta)\eta_k^2 &(3.46)\end{aligned}$$

where

$$\alpha_1^2(\delta,\beta) \coloneqq \Lambda - \beta\frac{\lambda\theta^2}{C_1\Lambda_1\eta_0^2}\delta, \quad \alpha_2^2(\delta,\beta) \coloneqq (1+\delta)\left(1-(1-\beta)\lambda\theta^2\right). \tag{3.47}$$

Choose δ small enough so that

$$\alpha^2 \coloneqq \max\{\alpha_1^2, \alpha_2^2\} < 1.$$

To ensure such a δ exists in light of the quasi-orthogonality constant $\Lambda > 1$ observe

$$\alpha_1^2 < 1 \text{ when } \delta > (\Lambda - 1)\frac{C_1\Lambda_1\eta_0^2}{\beta\lambda\theta^2}$$

and

$$\alpha_2^2 < 1 \text{ when } \delta < \left(1-(1-\beta)\lambda\theta^2\right)^{-1} - 1 = \frac{(1-\beta)\lambda\theta^2}{1-(1-\beta)\lambda\theta^2}$$

so to obtain an interval of positive measure where δ may be found we require

$$(\Lambda - 1)\frac{C_1\Lambda_1\eta_0^2}{\beta\lambda\theta^2} < \frac{(1-\beta)\lambda\theta^2}{1-(1-\beta)\lambda\theta^2}$$

placing a second constraint on the quasi-orthogonality constant

$$\Lambda < 1 + \frac{\lambda^2\theta^4\beta(1-\beta)}{C_1\Lambda_1\eta_0^2\left(1-(1-\beta)\lambda\theta^2\right)} \tag{3.48}$$

where $0 < \beta < 1$ and $\theta < 1$ may be chosen. In order to place bounds on the growth rate of the mesh, we further require $\theta < \theta_*$ given by (4.5) as discussed in section §4. □

Notice the choice of δ small enough to satisfy $\alpha^2 < 1$ is always possible, as each term may be independently driven below unity by a sufficiently small value of δ, so long as the quasi-orthogonality constant Λ is sufficiently close to one. For a discussion on the optimal contraction factor see Remark 4.3 in [4]; see also the discussion in [8].

3.6. **Convergence of GOAFEM.** We now derive a bound on error in the goal function.

Theorem 3.6 (GOAFEM functional convergence)**.** *Let the problem data satisfy Assumption 2.1 and the mesh satisfy Assumption 2.2. Let u the solution to* (1.3) *and z the solution to* (2.1)*. Let $\theta \in (0,1]$, and let $\{\mathcal{T}_k, \mathbb{V}_k, u_k, z_k\}_{k\geq 0}$ be the sequence of meshes, finite element spaces and discrete primal and dual solutions produced by GOAFEM. Let γ_p the constant γ from Theorem 3.5 applied to the primal problem* (2.22) *and γ_d the constant γ from Theorem 3.5 applied to the dual* (2.23)*. Then for constant $\alpha < 1$ as determined by Theorem 3.5*

$$\begin{aligned}|g(u) - g(u_k)| \leq 2\left\{\alpha^{2k}\left(|\!|\!| u - u_0 |\!|\!|^2 + \gamma_p\eta_0^2(u_0,\mathcal{T}_0)\right) - \gamma_p\eta_k^2\right\}^{1/2} \\ \times\left\{\alpha^{2k}\left(|\!|\!| z - z_0 |\!|\!|^2 + \gamma_d\zeta_0^2(z_0,\mathcal{T}_0)\right) - \gamma_d\zeta_k^2\right\}^{1/2}.\end{aligned}$$

Proof. On the primal side for all $v_k \in \mathbb{V}_k$

$$a(u-u_k, v_k) = a(u, v_k) - a(u_k, v_k) = f(v_k) - f(v_k) = 0,$$

the primal Galerkin orthogonality property. On the dual side, $g(u) = a^*(z,u)$ and $g(u_k) = a^*(z, u_k,)$ so that

$$\begin{aligned} g(u) - g(u_k) &= a^*(z, u - u_k) \\ &= a(u - u_k, z) \\ &= a(u-u_k, z - z_k). \end{aligned} \tag{3.49}$$

Define an inner-product α by the symmetric part of $a(\,\cdot\,,\,\cdot\,)$

$$\alpha(v,w) = \langle A\nabla v, \nabla w\rangle + \langle cv, w\rangle,$$

then

$$|||v|||^2 = a(v,v) = \alpha(v,v),$$

and

$$a(v,w) = \alpha(v,w) + \langle b\cdot\nabla v, w\rangle.$$

Then as $\alpha(\cdot,\cdot)$ is a symmetric bilinear form on Hilbert space; it is an inner product and it induces a norm identical to the energy norm induced by $a(\,\cdot\,,\,\cdot\,)$. As such we may apply the Cauchy-Schwarz inequality [6] to α and we're left to handle the convection term.

$$\begin{aligned} a(u-u_k, z-z_k) &= \alpha(u-u_k, z-z_k) + \langle b\cdot\nabla(u-u_k), z-z_k\rangle \\ &\le |||u-u_k|||\,|||z-z_k||| + \langle b\cdot\nabla(u-u_k), z-z_k\rangle. \end{aligned} \tag{3.50}$$

By Hölder's inequality followed by a duality estimate as in §6 on the dual error and coercivity on the primal,

$$\langle b\cdot\nabla(u-u_k), z-z_k\rangle \le \|b\|_{L_\infty} C_* h_0^s \mu_0^{-1/2} |||z-z_k|||\,|||u-u_k|||. \tag{3.51}$$

Recalling $\bar\Lambda = \|b\|_{L_\infty} C_* h_0^s \mu_0^{-1/2}$

$$a(u-u_k, z-z_k) \le |||u-u_k|||\,|||z-z_k||| + \bar\Lambda |||u-u_k|||\,|||z-z_k|||. \tag{3.52}$$

Under assumption (3.6) ($\bar\Lambda < 1$) on the initial mesh and from (3.49),

$$|g(u)-g(u_k)| = |a(u-u_k, z-z_k)| \le 2|||u-u_k|||\,|||z-z_k|||. \tag{3.53}$$

From 3.5 there is an $\alpha < 1$ such that for the primal problem with estimator η_k

$$|||u-u_{k+1}|||^2 \le \alpha^2\left(|||u-u_k|||^2 + \gamma_p\eta_k^2\right) - \gamma_p\eta_{k+1}^2 \tag{3.54}$$

and for the dual problem with estimator ζ_k

$$|||z-z_{k+1}|||^2 \le \alpha^2\left(|||z-z_k|||^2 + \gamma_d\zeta_k^2\right) - \gamma_d\zeta_{k+1}^2. \tag{3.55}$$

Iterating, we have from (3.54) and (3.55)

$$|||u-u_k|||^2 + \gamma_p\eta_k^2 \le \alpha^{2k}\left(|||u-u_0|||^2 + \gamma_p\eta_0^2\right) \tag{3.56}$$

$$|||z-z_k|||^2 + \gamma_d\zeta_k^2 \le \alpha^{2k}\left(|||z-z_0|||^2 + \gamma_d\zeta_0^2\right). \tag{3.57}$$

From (3.53), (3.56) and (3.57) obtain the contraction of error in quantity of interest

$$\begin{aligned} |g(u)-g(u_k)| \le 2&\left\{\alpha^{2k}\left(|||u-u_0|||^2 + \gamma_p\eta_0^2(u_0,\mathcal{T}_0)\right) - \gamma_p\eta_k^2\right\}^{1/2} \\ &\times\left\{\alpha^{2k}\left(|||z-z_0|||^2 + \gamma_d\zeta_0^2(z_0,\mathcal{T}_0)\right) - \gamma_d\zeta_k^2\right\}^{1/2}, \end{aligned} \tag{3.58}$$

or more simply

$$\begin{aligned} |g(u)-g(u_k)| + \gamma_p\eta_k^2 + \gamma_d\zeta_k^2 &\le \alpha^{2k}\left(|||u-u_0|||^2 + \gamma_p\eta_0^2(u_0,\mathcal{T}_0)\right. \\ &\qquad \left. + |||z-z_0|||^2 + \gamma_d\zeta_0^2(z_0,\mathcal{T}_0)\right) \qquad (3.59) \\ &= \alpha^{2k} Q_0^2 \qquad (3.60) \end{aligned}$$

with Q_0 the quasi-error on the initial mesh.

□

4. QUASI-OPTIMAL CARDINALITY OF GOAFEM

In this section we establish the quasi-optimality of GOAFEM. The result in §4.5 follows from bounding the cardinality of the marked set for each of the primal and dual problems at each iteration as shown in Lemma 4.9. This is achieved by assuming the primal and dual solutions belong to appropriate approximation classes as discussed in §4.4, the optimality assumptions addressed in §4.2, and the supporting results below. Under the optimality assumptions, the error-indicator as an upper-bound on energy-error as shown in §4.1 and a bound for the oscillation term as the mesh is refined as shown in §4.2, a suitable reduction in global error between two consecutive iterations implies the respective refinement set satisfies the Dörfler property. We address the effect of quasi-orthogonality on the necessary reduction to achieve this result.

The estimator as global lower bound on total error in §4.1 is used to relate the total-error to the quasi-error in §4.5, connecting the contraction property for the quasi-error established in §3 to the quasi-optimality of the total error in §4.3 which shows the total error satisfies Céa's Lemma.

4.1. **Estimator as global lower bound and localized upper bound.** We start with two fairly standard results that will be needed in the complexity analysis. The *global lower bound* may be found in [9] Lemma 3.1 and a similar result in [10] Proposition 4.3 and Corollary 4.4. The *localized upper bound* is established in [4] Lemma 3.6.

Lemma 4.1 (Global lower bound)**.** *Let the problem data satisfy Assumption 2.1 and the mesh satisfy Assumption 2.2. Let $\mathcal{T}_1, \mathcal{T}_2 \in \mathbb{T}$ and $\mathcal{T}_2 \geq \mathcal{T}_1$ a full refinement. Let $u_k \in \mathbb{V}_k$ the solution to* (2.22)*, $k = 1, 2$. Then there is a global constant $c_2 > 0$ such that*

$$c_2 \eta_1^2(u_1, \mathcal{T}_1) \leq |||u - u_1|||^2 + \text{osc}_1^2(u_1, \mathcal{T}_1). \tag{4.1}$$

Lemma 4.2 (Localized upper bound)**.** *Let the problem data satisfy Assumption 2.1 and the mesh satisfy conditions (1) and (2) of Assumption 2.2. Let $\mathcal{T}_1, \mathcal{T}_2 \in \mathbb{T}$ with $\mathcal{T}_2 \geq \mathcal{T}_1$. Let $\mathcal{R} \coloneqq \mathcal{R}_{\mathcal{T}_1 \to \mathcal{T}_2}$ the set of refined elements. Let $u_k \in \mathbb{V}_k$ the solution to* (2.22)*, $k = 1, 2$. Then there is a global constant C_1 with*

$$|||u_2 - u_1|||^2 \leq C_1 \eta_1^2(u_1, \mathcal{R}). \tag{4.2}$$

4.2. **Optimality assumptions and optimal marking.** In this section we consider the assumptions on marking parameter θ and the marking strategy which allow us to characterize the growth of the adaptive mesh at each iteration with respect to the total error in 4.5.

We first consider oscillation on the refined mesh, following closely [4], Corollary 3.5.

Lemma 4.3 (Oscillation on refined mesh)**.** *Let the problem data satisfy Assumption 2.1 and the mesh satisfy condition (1) of Assumption 2.2. Let $\mathcal{T}_1, \mathcal{T}_2 \in \mathbb{T}$ with $\mathcal{T}_2 \geq \mathcal{T}_1$. Let $\Lambda_2 = \bar{\Lambda}_2^2 m_{\mathcal{E}}^{-2}$ with $\bar{\Lambda}_2$ from* (3.13)*. Then for all $v_1 \in \mathbb{V}_1$ and $v_2 \in \mathbb{V}_2$*

$$\text{osc}_1^2(v_1, \mathcal{T}_1 \cap \mathcal{T}_2) \leq 2\text{osc}_2^2(v_2, \mathcal{T}_1 \cap \mathcal{T}_2) + 2\Lambda_2 \text{osc}_0^2 |||v_1 - v_2|||^2, \tag{4.3}$$

where $\text{osc}_0^2 \coloneqq \text{osc}_{\mathcal{T}_0}^2(D, \mathcal{T}_0)$.

Proof. For all elements T in the intersection $T \in \mathcal{T}_1 \cap \mathcal{T}_2$

$$\text{osc}_1(v_1, T) = \text{osc}_2(v_1, T).$$

Applying this, $v_1 \in \mathbb{V}_1 \subset \mathbb{V}_2$ and $\text{osc}_j(D, T) \leq \text{osc}_0(D, T),\ j = 1, 2$, we have from (3.13)

$$\text{osc}_2(v_1, T) \leq \text{osc}_2(v_2, T) + \bar{\Lambda}_2 \text{osc}_0 \|v - w\|_{H^1(T)}^2.$$

Squaring and applying Young's inequality with $\varepsilon = 1$ yields

$$\text{osc}_1^2(v_1, T) \le 2\text{osc}_2^2(v_2, T) + 2\bar{\Lambda}_2^2 \text{osc}_0^2 \|v_1 - v_2\|_{H^1(T)}^2. \tag{4.4}$$

Summing over all $T \in \mathcal{T}_1 \cap \mathcal{T}_2$ and bounding the norm over $\mathcal{T}_1 \cap \mathcal{T}_2$ to the entire domain to apply the coercivity estimate (2.9)

$$\text{osc}_1^2(v_1, \mathcal{T}_1 \cap \mathcal{T}_2) \le 2\text{osc}_2^2(v_2, \mathcal{T}_1 \cap \mathcal{T}_2) + 2\Lambda_2 \text{osc}_0^2 |||v_1 - v_2|||^2.$$

□

We now discuss some basic assumptions for complexity analysis. The optimality assumptions follow those found in [4] with modifications in (4.5) to account for the nonsymmetric problem, the continuity of A and the goal-oriented method.

Assumption 4.4 (Optimality assumptions)**.** *Assume the following conditions.*

1) *The marking parameter θ satisfies $\theta \in (0, \theta_*)$ with*

$$\theta_* = \frac{c_2}{1 + C_1(1 + \bar{\Lambda} + 2\Lambda_2 \text{osc}_0^2)}, \text{ with } \text{osc}_0 = \text{osc}_0^2(D, \mathcal{T}_0) \tag{4.5}$$

   and $\bar{\Lambda}$ given by (3.6)*. As the data oscillation given by* (2.35) *is identical for the primal and dual problems and the other constants depend only on global data, θ_* may be assumed the same for both the primal an dual problems.*
2) *A marked set $\mathcal{M}_k$ of optimal cardinality (up to a factor of two) is selected (see* [10]*).*
3) *The distribution of refinement edges on $\mathcal{T}_0$ satsifies condition (b) of section 4 in* [12]*.*

We now consider a basic result on optimal marking. This lemma is a variation of Lemma 5.9 in [4], modified to use quasi-orthogonality 3.1 rather than Galerkin orthogonality.

Lemma 4.5 (Optimal marking)**.** *Let the problem data satisfy Assumption 2.1 and the mesh satisfy Assumption 2.2. Let $\mathcal{T}_1, \mathcal{T}_2 \in \mathbb{T}$. Let $u_k \in \mathbb{V}_k$ the solution to* (2.22)*, $k = 1, 2$. Let the marking parameter θ satisfy condition (1) of Assumption 4.4.*

Let $\mathcal{T}_2 \ge \mathcal{T}_1$ satisfy

$$|||u - u_2|||^2 + \text{osc}_2^2 \le \frac{\mu}{\alpha}\left(|||u - u_1|||^2 + \text{osc}_1^2\right) \tag{4.6}$$

which implies

$$\alpha|||u - u_2|||^2 + \text{osc}_2^2 \le \mu\left(|||u - u_1|||^2 + \text{osc}_1^2\right) \tag{4.7}$$

for $\mu := \frac{1}{2}(1 - \frac{\theta^2}{\theta_*^2})$ and $\alpha = (1 + \bar{\Lambda})$, $\bar{\Lambda} \in (0, 1)$ given by (3.6) *in the quasi-orthogonality argument and where*

$$\text{osc}_1 = \text{osc}_1(u_1, \mathcal{T}_1), \quad \text{osc}_2 = \text{osc}_2(u_2, \mathcal{T}_2), \quad \textit{and } \eta_1 = \eta_1(u_1, \mathcal{T}_1).$$

Then the set $\mathcal{R} := \mathcal{R}_{\mathcal{T}_1 \to \mathcal{T}_2}$ satisfies the Dörfler property

$$\eta_1(u_1, \mathcal{R}) \ge \theta \eta_1(u_1, \mathcal{T}_1).$$

Proof. (See [4] Lemma 5.9). As $0 < 2\mu < 1$, multiply inequality (4.1) by $1 - 2\mu$ to obtain

$$(1 - 2\mu)c_2\eta_1^2 \le |||u - u_1|||^2 + \text{osc}_1^2 - 2\mu\left(|||u - u_1|||^2 + \text{osc}_1^2\right).$$

Applying (4.7)

$$(1 - 2\mu)c_2\eta_1^2 \le |||u - u_1|||^2 - \alpha|||u - u_2|||^2 + \text{osc}_1^2 - 2\text{osc}_2^2.$$

Rearranging terms obtain

$$\text{osc}_1^2 - 2\text{osc}_2^2 \geq (1-2\mu)c_2\eta_1^2 + \alpha|\!|\!|u-u_2|\!|\!|^2 - |\!|\!|u-u_1|\!|\!|^2. \tag{4.8}$$

By the second quasi-orthogonality estimate (3.9)

$$(1+\bar{\Lambda})|\!|\!|u-u_2|\!|\!|^2 - |\!|\!|u-u_1|\!|\!|^2 \geq -(1+\bar{\Lambda})|\!|\!|u_1-u_2|\!|\!|^2$$

where $0 < \bar{\Lambda} < 1$. Applying (4.2)

$$(1+\bar{\Lambda})|\!|\!|u-u_2|\!|\!|^2 - |\!|\!|u-u_1|\!|\!|^2 \geq -(1+\bar{\Lambda})C_1\eta_1^2(u_1,\mathcal{R}). \tag{4.9}$$

Combining (4.9) with (4.8) obtain

$$\text{osc}_1^2 - 2\text{osc}_2^2 \geq (1-2\mu)c_2\eta_1^2 - (1+\bar{\Lambda})C_1\eta_1^2(u_1,R). \tag{4.10}$$

For refined elements $T \in \mathcal{R}$ use the dominance of the estimator over the oscillation

$$\text{osc}_1^2(u_1,T) \leq \eta_1^2(u_1,T).$$

For elements $T \in \mathcal{T}_1 \cap \mathcal{T}_2$ (4.3) yields

$$\text{osc}_1^2(u_1,\mathcal{T}_1\cap\mathcal{T}_2) - 2\text{osc}_2^2(u_2,\mathcal{T}_1\cap\mathcal{T}_2) \leq 2\Lambda_2\text{osc}_0^2|\!|\!|u_1-u_2|\!|\!|^2.$$

Then

$$\text{osc}_1^2(u_1,\mathcal{T}_1) - 2\text{osc}_2^2(u_2,\mathcal{T}_2) \leq \eta_1^2(u_1,\mathcal{R}) + 2\Lambda_2\text{osc}_0^2|\!|\!|u_1-u_2|\!|\!|^2.$$

Applying (4.2) to the last term

$$\text{osc}_1^2(u_1,\mathcal{T}_1) - 2\text{osc}_2^2(u_2,\mathcal{T}_2) \leq (1+2C_1\Lambda_2\text{osc}_0^2)\eta_1^2(u_1,\mathcal{R}). \tag{4.11}$$

Rearranging terms in (4.11) and applying (4.10)

$$\eta_1^2(u_1,\mathcal{R}) \geq \frac{(1-2\mu)c_2\eta_1^2 - (1+\bar{\Lambda})C_1\eta_1^2(u_1,R)}{(1+2C_1\Lambda_2\text{osc}_0^2)}.$$

Combining like terms obtain

$$\eta_1^2(u_1,\mathcal{R}) \geq \frac{(1-2\mu)c_2}{1+C_1(1+\bar{\Lambda}+2\Lambda_2\text{osc}_0^2)}\eta_1^2.$$

Applying the definitions of μ and θ_* obtain the result

$$\eta_1^2(u_1,\mathcal{R}) \geq \theta^2\eta_1^2.$$

□

Due to the use of quasi-orthogonality, the required assumption (4.7) is stronger than

$$|\!|\!|u-u_2|\!|\!|^2 + \text{osc}_2^2 \leq \mu\left(|\!|\!|u-u_1|\!|\!|^2 + \text{osc}_1^2\right)$$

the condition in [4] for the symmetric problem, but it is also weaker than

$$|\!|\!|u-u_2|\!|\!|^2 + \text{osc}_2^2 \leq \frac{\mu}{\alpha}\left(|\!|\!|u-u_1|\!|\!|^2 + \text{osc}_1^2\right)$$

where $\alpha = 1+\bar{\Lambda} > 1$, formally similar to the symmetric estimate. We may impose this stronger condition for ease of analysis, however in practice this says that the increase in error-reduction we require of the finer mesh needs only come from the energy-norm error, not the oscillation.

We recall a standard result on the mesh overlap, see [4] Lemma 3.7.

Lemma 4.6 (Overlay of meshes)**.** *Let the mesh satisfy condition (1) of Assumption 2.2. Let $\mathcal{T}_1, \mathcal{T}_2 \in \mathbb{T}$. Then the overlay $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2$ is conforming and satisfies*

$$\#\mathcal{T} \leq \#\mathcal{T}_1 + \#\mathcal{T}_2 - \#\mathcal{T}_0.$$

4.3. **Quasi-optimality of total errror.** We show the total error satisfies Céa's Lemma; e.g. see [4] Lemma 5.2. This version appropriate for the non-symmetric problem relies quasi-orthogonality 3.1 rather than Galerkin orthogonality.

Theorem 4.7 (Quasi-optimality of total error)**.** *Let the problem data satisfy Assumption 2.1 and the mesh satisfy Assumption 2.2. Let $\mathcal{T}_1 \in \mathbb{T}$. Let u the solution of* (1.3) *and $u_1 \in \mathbb{V}_1$ the solution of* (2.22)*. Then there is a constant C_D depending on the initial mesh $\mathcal{T}_0$ and the problem data D such that*

$$|\!|\!| u - u_1 |\!|\!|^2 + \text{osc}_1^2(u_1, \mathcal{T}_1) \leq C_D \inf_{v \in \mathbb{V}_1} \left(|\!|\!| u - v |\!|\!|^2 + \text{osc}_1^2(v, \mathcal{T}_1) \right). \tag{4.12}$$

Proof. For $\varepsilon > 0$ choose $v_\varepsilon \in \mathbb{V}_1$ with

$$|\!|\!| u - v_\varepsilon |\!|\!|^2 + \text{osc}_1^2(v_\varepsilon, \mathcal{T}_1) \leq (1+\varepsilon) \inf_{v \in \mathbb{V}_1} \left(|\!|\!| u - v |\!|\!|^2 + \text{osc}_1^2(v, \mathcal{T}_1) \right).$$

By (4.3) with $\mathcal{T}_2 = \mathcal{T}_1$ obtain

$$\text{osc}_1^2(v_1, \mathcal{T}_1) \leq 2\text{osc}_1^2(v_\varepsilon, \mathcal{T}_1) + 2\Lambda_2 \text{osc}_0^2 |\!|\!| u_1 - v_\varepsilon |\!|\!|^2. \tag{4.13}$$

By the same reasoning as (3.1) obtain

$$|\!|\!| u - u_1 |\!|\!|^2 + |\!|\!| u_1 - v_\varepsilon |\!|\!|^2 \leq \Lambda |\!|\!| u - v_\varepsilon |\!|\!|^2$$

which implies

$$|\!|\!| u - u_1 |\!|\!|^2 \leq \Lambda |\!|\!| u - v_\varepsilon |\!|\!|^2 \text{ and } |\!|\!| u_1 - v_\varepsilon |\!|\!|^2 \leq \Lambda |\!|\!| u - v_\varepsilon |\!|\!|^2. \tag{4.14}$$

From (4.13) and (4.14) obtain

$$\begin{aligned} |\!|\!| u - u_1 |\!|\!|^2 + \text{osc}_1^2(u_1, \mathcal{T}_1) &\leq \Lambda |\!|\!| u - v_\varepsilon |\!|\!|^2 + 2\text{osc}_1^2(v_\varepsilon, \mathcal{T}_1) + 2\Lambda_2 \text{osc}_0^2 |\!|\!| u_1 - v_\varepsilon |\!|\!|^2 \\ &\leq \Lambda \left(1 + 2\Lambda_2 \text{osc}_0^2\right) |\!|\!| u - v_\varepsilon |\!|\!|^2 + 2\text{osc}_1(v_\varepsilon, \mathcal{T}_1). \end{aligned}$$

Set $C_D := \max\{2, \Lambda\,(1 + 2\Lambda_2 \text{osc}_0^2)\}$ then

$$\begin{aligned} |\!|\!| u - u_1 |\!|\!|^2 + \text{osc}_1^2(u_1, \mathcal{T}_1) &\leq C_D \left(|\!|\!| u - v_\varepsilon |\!|\!|^2 + \text{osc}_1(v_\varepsilon, \mathcal{T}_1) \right) \\ &\leq C_D (1+\varepsilon) \inf_{v \in \mathbb{V}_1} \left(|\!|\!| u - v |\!|\!|^2 + \text{osc}_1^2(v, \mathcal{T}_1) \right). \end{aligned}$$

Letting $\varepsilon \to 0$ establishes the result. □

4.4. **Approximation classes and approximation property.** For problem with solution, forcing function and data (u, f, D) and dual problem (z, g, D^*), membership in an appropriate approximation class says the solution u (respectively z) may be approximated within a given tolerance by finite element approximation while the cardinality of the mesh required to achieve the tolerance satisfies (4.18).

For $N > 0$ let $\mathbb{T}_N$ the set of conforming triangulations generated from the initial mesh $\mathcal{T}_0$ such that the increase in cardinality is at most N

$$\mathbb{T}_N := \{\mathcal{T} \in \mathbb{T} \mid \#\mathcal{T} - \#\mathcal{T}_0 \leq N\}.$$

For $s > 0$ define the standard approximation classes for solutions based on the energy error

$$\mathcal{A}_s := \left\{ v \in \mathbb{V} \mid \sup_{N>0} (N^s \inf_{\mathcal{T} \in \mathbb{T}_N} \inf_{v_\mathcal{T} \in \mathbb{V}_\mathcal{T}} |\!|\!| v - v_\mathcal{T} |\!|\!| < \infty \right\} \tag{4.15}$$

and for L_2 data

$$\bar{\mathcal{A}}_s := \left\{ g \in L_2(\Omega) \mid \sup_{N>0} (N^s \inf_{\mathcal{T} \in \mathbb{T}_N} \| h(g - \Pi_{2n-2}^2 g) \|_{L_2(\Omega)}) < \infty \right\}. \tag{4.16}$$

Define a measure of approximation based on the *total error*

$$\sigma(N; v, f, D) := \inf_{\mathcal{T}\in\mathbb{T}_N} \inf_{v_\mathcal{T}\in V_\mathcal{T}} \left(|||v - v_\mathcal{T}|||^2 + \text{osc}^2_\mathcal{T}(v_\mathcal{T}, \mathcal{T})\right)^{\frac{1}{2}}$$

and denote the total error of $v_\mathcal{T} \in \mathbb{V}_\mathcal{T}$ by

$$E(v, \mathcal{T}) := \left(|||v - v_\mathcal{T}|||^2 + \text{osc}^2_\mathcal{T}(v_\mathcal{T}, \mathcal{T})\right)^{1/2}$$

and the approximation class based on the total error for $s > 0$

$$\mathbb{A}_s := \left\{ (v, f, D) \,\middle|\, |v, f, D|_s := \sup_{N>0} (N^s \sigma(N; v, f, D)) < \infty \right\}. \tag{4.17}$$

See [4] Lemma 5.3 and Lemma 5.4 for a discussion on the relation between the classes $\mathbb{A}_s, \mathcal{A}_s$ and $\bar{\mathcal{A}}_s$. The results in this paper are developed with respect to the class $\mathbb{A}_s$ based on the total error.

Membership of the primal and dual solutions in the approximation classes $\mathbb{A}_s$ and $\mathbb{A}_t$ is applied via the use of the two properties discussed in this section.

Lemma 4.8 (Approximation property)**.** *Let the mesh satisfy condition (1) of Assumption 2.2. Let u the solution to* (1.3)*. Assume $u \in \mathbb{A}_s$ and $\sigma(1; u, f, D) > 0$. Then given $\varepsilon > 0$ there is a global constant C depending only on the initial mesh $\mathcal{T}_0$ and the problem data D, a partition $\mathcal{T}_\varepsilon \in \mathcal{T}$ and a $v_\varepsilon \in \mathbb{V}_{\mathcal{T}_\varepsilon}$ such that*

$$C|u, f, D|_s \geq (\#\mathcal{T}_0 - \mathcal{T}_\varepsilon)^s \varepsilon \tag{4.18}$$

$$E(v_\varepsilon, \mathcal{T}_\varepsilon) \leq \varepsilon. \tag{4.19}$$

Proof. By (4.17) and property of the supremum, for any $N > 0$

$$|u, f, D|_s \geq N^s \sigma(N; u, f, D) \tag{4.20}$$

where $N = \#\mathcal{T} - \#\mathcal{T}_0$. Given $\varepsilon > 0$ consider all $N > 0$ such that $\sigma(N; u, f, D) \geq \varepsilon$. If there is no such N, let $N_\varepsilon = 1$. By (4.20)

$$|u, f, D|_s \geq \sigma_1 = \frac{\sigma_1}{\varepsilon}\varepsilon \text{ where } \sigma_1 := \sigma(1; u, f, D).$$

Applying the assumption $\sigma(1; u, f, D) > 0$

$$\frac{\varepsilon}{\sigma_1}|u, f, D|_s \geq \varepsilon$$

establishing (4.18) with $C = \varepsilon/\sigma_1$. Also

$$\sigma(1; u, f, D) = \inf_{\mathcal{T}\in\mathbb{T}_1} \inf_{v\in\mathbb{V}_{\mathcal{T}_1}} E(v, \mathcal{T}) < \varepsilon$$

so there is $\mathcal{T}_\varepsilon \in \mathbb{T}_1$ and $v_\varepsilon \in \mathbb{V}_{\mathcal{T}_\varepsilon}$ so that $E(v_\varepsilon, \mathcal{T}_\varepsilon) \leq \varepsilon$ establishing (4.19). Otherwise, there is $N > 0$ with $\sigma(N; u, f, D) \geq \varepsilon$. As the infimum over the total error goes to zero as $N \to \infty$ this holds for finitely many N so define

$$K := \max\{N > 0 \,|\, \sigma(N; u, f, D) \geq \varepsilon\}. \tag{4.21}$$

By (4.20) and (4.21)

$$|u, f, D|_s \geq K^s \sigma(K; u, f, D) \geq K^s \varepsilon. \tag{4.22}$$

Let $N_\varepsilon = 2K$.

$$|u, f, D|_s \geq K^s \varepsilon = 2^{-s} N_\varepsilon^s \varepsilon \implies C|u, f, D|_s \geq N_\varepsilon^s \varepsilon$$

with $C = 2^s$ establishing (4.18). By (4.21) and property of the infimum with $N_\varepsilon > K$

$$\sigma(N_\varepsilon; u, f; D) = \inf_{\mathcal{T}\in\mathbb{T}_{N_\varepsilon}} \inf_{v\in\mathbb{V}_\varepsilon} E(v, \mathcal{T}) \leq \inf_{\mathcal{T}\in\mathbb{T}_{N_K}} \inf_{v\in\mathbb{V}_\varepsilon} E(v, \mathcal{T}) < \varepsilon$$

implying $E(v_\varepsilon, \mathcal{T}_\varepsilon) \le \varepsilon$ for some $\mathcal{T}_\varepsilon \in \mathbb{T}_\varepsilon$ and a $v_\varepsilon \in \mathbb{V}_{\mathcal{T}_\varepsilon}$ establishing (4.19). □

4.5. **Cardinality of $\mathcal{M}_k$ and quasi-optimality of the mesh.** The results on the cardinality of $\mathcal{M}_k$ and quasi-optimality are variations on [4] Lemma 5.10 and Theorem 5.11. Here we address the goal-oriented method discussed in 2.3.

Lemma 4.9 (Cardinality of $\mathcal{M}_k$)**.** *Let the problem data satisfy Assumption 2.1 and the mesh satisfy Assumption 2.2. Assume conditions (1) and (2) of Assumption 4.4. Let u the solution of* (1.3) *and z the solution of* (2.1)*. Let $\{\mathcal{T}_k, \mathbb{V}_k, u_k, z_k\}_{k\ge 0}$ the sequence of meshes, finite element spaces and discrete primal and dual solutions produced by GOAFEM. If $(u, f, D) \in \mathbb{A}_s$ and $(z, g, D^*) \in \mathbb{A}_t$ we have*

$$\#\mathcal{M}_k \le 2C \Bigg\{ (1+\bar\Lambda)^{1/2s} \left(1 - \frac{\theta^2}{\theta_*^2}\right)^{-1/2s} |u, f, D|_s^{1/s} C_D^{1/2s} E_k^{-1/s}(u_k, \mathcal{T}_k) + (1+\bar\Lambda)^{1/2t} \left(1 - \frac{\theta^2}{\theta_*^2}\right)^{-1/2t} |z, g, D^*|_t^{1/t} C_D^{1/2t} E_k^{-1/t}(z_k, \mathcal{T}_k) \Bigg\} \tag{4.23}$$

where C_D is the constant from (4.12) *and the total errors in the primal and dual problems*

$$E_k^2(u_k, \mathcal{T}_k) := \|u - u_k\|^2 + \mathrm{osc}_k^2(u_k, \mathcal{T}_k)$$
$$E_k^2(z_k, \mathcal{T}_k) := \|z - z_k\|^2 + \mathrm{osc}_k^2(z_k, \mathcal{T}_k).$$

Proof. Set $\tilde\mu = \frac{1}{2}\left(1 - \frac{\theta^2}{\theta_*^2}\right)(1+\bar\Lambda)^{-1}$ with $\bar\Lambda$ given by (3.6).

$$\varepsilon_p^2 := \tilde\mu C_D^{-1} E_k^2(u_k, \mathcal{T}_k), \text{ and } \varepsilon_d^2 := \tilde\mu C_D^{-1} E_k^2(z_k, \mathcal{T}_k).$$

As $(u, f, D) \in \mathbb{A}_s$, by the properties in section 4.4 there is a $\mathcal{T}_p \in \mathbb{T}$ and a $v_p \in \mathbb{V}_{\mathcal{T}_p}$ such that

$$\#\mathcal{T}_p - \#\mathcal{T}_0 \le C|u, f, D|_s^{1/s} \varepsilon_p^{-1/s} \tag{4.24}$$
$$\|u - v_p\|^2 + \mathrm{osc}_{\mathcal{T}_p}^2(v_p, \mathcal{T}_p) \le \varepsilon_p^2. \tag{4.25}$$

Similarly for $(z, g, D^*) \in \mathbb{A}_t$, there is a $\mathcal{T}_d \in \mathbb{T}$ and a $w_d \in \mathbb{V}_{\mathcal{T}_d}$ such that

$$\#\mathcal{T}_d - \#\mathcal{T}_0 \le C|z, g, D^*|_t^{1/t} \varepsilon_d^{-1/t} \tag{4.26}$$
$$\|z - w_d\|^2 + \mathrm{osc}_{\mathcal{T}_d}^2(w_d, \mathcal{T}_d) \le \varepsilon_d^2. \tag{4.27}$$

Let $\mathcal{T}_2 := \mathcal{T}_k \oplus (\mathcal{T}_p \oplus \mathcal{T}_d)$ as in Lemma 4.6. Let $u_2 \in \mathbb{V}_2$ the Galerkin solution to (2.22) and $z_2 \in \mathbb{V}_2$ the respective solution to (2.23) . See there is a reduction in the total error by a factor of $\tilde\mu$ from u_k to u_2 (respectively z_k to z_2). Since $\mathcal{T}_2 \ge \mathcal{T}_p$ by Theorem 4.7, monotonicity of infimum over total error and (4.25)

$$\begin{aligned} \|u - u_2\|^2 + \mathrm{osc}_2^2(u_2, \mathcal{T}_2) &\le C_D \inf_{v \in \mathbb{V}_2} \left(\|u - v\|^2 + \mathrm{osc}_2^2(v, \mathcal{T}_2)\right) \\ &\le C_D \varepsilon_p^2 \\ &= \tilde\mu \left(\|u - u_k\|^2 + \mathrm{osc}_k^2(u_k, \mathcal{T}_k)\right). \end{aligned} \tag{4.28}$$

Similarly for the dual problem

$$\|z - z_2\|^2 + \mathrm{osc}_2^2(z_2, \mathcal{T}_2) \le \tilde\mu \left(\|z - z_k\|^2 + \mathrm{osc}_k^2(z_k, \mathcal{T}_k)\right). \tag{4.29}$$

This satisfies the hypothesis (4.6) in each problem so applying 4.2 the refining subset $\mathcal{R} := \mathcal{R}_{\mathcal{T}_k \to \mathcal{T}_2} \subset \mathcal{T}_k$ satisfies the Dörfler property for $\theta \le \theta_*$. The marking procedure selects a subset for marking $\mathcal{M}_k \subset \mathcal{T}_k$ of minimal cardinality up to a factor of two so that by Lemma 4.6

$$\#\mathcal{M}_k \leq 2\#\mathcal{R} \leq 2(\#\mathcal{T}_2 - \#\mathcal{T}_k) \leq 2\left\{(\#\mathcal{T}_p - \#\mathcal{T}_0) + (\#\mathcal{T}_d - \#\mathcal{T}_0)\right\}. \tag{4.30}$$

By (4.30), (4.24), the definition of ε_p and ε_d, (4.28) and the definition of μ

$$\begin{aligned}
\#\mathcal{M}_k &\leq 2\left\{(\#\mathcal{T}_p - \#\mathcal{T}_0) + (\#\mathcal{T}_d - \#\mathcal{T}_0)\right\} \\
&\leq 2C\left\{|u,f,D|_s^{1/s}\varepsilon_p^{-1/s} + |z,g,D^*|_t^{1/t}\varepsilon_d^{-1/t}\right\} \\
&= 2C\left\{(1+\bar{\Lambda})^{1/2s}\left(1-\frac{\theta^2}{\theta_*^2}\right)^{-1/2s}|u,f,D|_s^{1/s}C_D^{1/2s}E_k^{-1/s}(u_k,\mathcal{T}_k)\right. \\
&\qquad\left. +(1+\bar{\Lambda})^{1/2t}\left(1-\frac{\theta^2}{\theta_*^2}\right)^{-1/2t}|z,g,D^*|_t^{1/t}C_D^{1/2t}E_k^{-1/t}(z_k,\mathcal{T}_k).\right\}
\end{aligned}$$

□

Theorem 4.10 (Quasi-optimality)**.** *Let the problem data satisfy Assumption 2.1 and the mesh satisfy Assumption 2.2. Let Assumption 4.4 be satisfied by GOAFEM. Let u the solution of* (1.3) *and z the solution of* (2.1)*. Let $\{\mathcal{T}_k, \mathbb{V}_k, u_k, z_k\}_{k\geq 0}$ the sequence of meshes, finite element spaces and discrete primal and dual solutions produced by GOAFEM. Let $(u,f,D) \in \mathbb{A}_s$ and $(z,g,D^*) \in \mathbb{A}_t$. Then*

$$\begin{aligned}
\#\mathcal{T}_k - \#\mathcal{T}_0 \leq S(\theta)&\left\{M_p\left(1+\frac{\gamma_p}{c_2}\right)^{1/2s}Q_k^{-1/s}(u_k,\mathcal{T}_k)\right. \\
&\left. + M_d\left(1+\frac{\gamma_d}{c_2}\right)^{1/2t}Q_k^{-1/t}(z_k,\mathcal{T}_k)\right\}.
\end{aligned}$$

Proof. Let the total error in primal and dual problems $E_k(u_k,\mathcal{T}_k)$ and $E_k(z_k,\mathcal{T}_k)$ as in Lemma 4.9. Denote the quasi-error in each problem by

$$\begin{aligned}
Q_k^2(u_k,\mathcal{T}_k) &:= |||u-u_k|||^2 + \gamma_p\eta_k^2(u_k,\mathcal{T}_k), \\
Q_k^2(z_k,\mathcal{T}_k) &:= |||z-z_k|||^2 + \gamma_d\zeta_k^2(z_k,\mathcal{T}_k).
\end{aligned}$$

As shown in [2] Theorem 2.4 there is a global constant C_f which satisfies

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \leq C_f\sum_{j=0}^{k-1}\#\mathcal{M}_j \quad \text{for all } k \geq 1$$

and by (4.23)

$$\begin{aligned}
\#\mathcal{M}_k \leq 2C&\left\{(1+\bar{\Lambda})^{1/2s}\left(1-\frac{\theta^2}{\theta_*^2}\right)^{-1/2s}|u,f,D|_s^{1/s}C_D^{1/2s}E_k^{-1/s}(u_k,\mathcal{T}_k)\right. \\
&\left. +(1+\bar{\Lambda})^{1/2t}\left(1-\frac{\theta^2}{\theta_*^2}\right)^{-1/2t}|z,g,D^*|_t^{1/t}C_D^{1/2t}E_k^{-1/t}(z_k,\mathcal{T}_k)\right\}
\end{aligned}$$

then we have

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \leq M_p\sum_{j=0}^{k-1}E_k(u_k,\mathcal{T}_k)^{-1/s} + M_d\sum_{j=0}^{k-1}E_k(z_k,\mathcal{T}_k)^{-1/t} \tag{4.31}$$

with the constants

$$M_p := 2C_f C(1+\bar{\Lambda})^{1/2s}\left(1-\frac{\theta^2}{\theta_*^2}\right)^{-1/2s}|u,f,D|_s^{1/s}C_D^{1/2s}$$

$$M_d := 2C_f C(1+\bar{\Lambda})^{1/2t}\left(1-\frac{\theta^2}{\theta_*^2}\right)^{-1/2t}|z,g,D^*|_t^{1/t}C_D^{1/2t}.$$

From the domination of the error estimator over the oscillation and the lower bound on total error (4.1) we have the equivalence of the total-error and quasi-error

$$\begin{aligned}|\!|\!| u-u_j |\!|\!|^2 + \gamma_p \mathrm{osc}_j^2(u_j,\mathcal{T}_j) &\le |\!|\!| u-u_j |\!|\!|^2 + \gamma_p \eta_j^2(u_j,\mathcal{T}_j)\\ &\le \left(1+\frac{\gamma_p}{c_2}\right)E^2(u_j,\mathcal{T}_j). \end{aligned} \tag{4.32}$$

or

$$E_j^{-1/s}(u_j,\mathcal{T}_j) \le \left(1+\frac{\gamma_p}{c_2}\right)^{1/2s} Q_j^{-1/s}(u_j,\mathcal{T}_j) \tag{4.33}$$

and similarly for the dual problem

$$E_j^{-1/t}(z_j,\mathcal{T}_j) \le \left(1+\frac{\gamma_d}{c_2}\right)^{1/2t} Q_j^{-1/t}(z_j,\mathcal{T}_j). \tag{4.34}$$

By the contraction result on the quasi-error (3.37) for $0 \le j \le k-1$

$$Q_k^2(u_k,\mathcal{T}_k) \le \alpha^{2(k-j)}Q_j^2(u_j,\mathcal{T}_j) \quad \text{and} \quad Q_k^2(z_k,\mathcal{T}_k) \le \alpha^{2(k-j)}Q_j^2(z_j,\mathcal{T}_j). \tag{4.35}$$

Putting together (4.31), (4.33) and (4.35) obtain

$$\begin{aligned}\#\mathcal{T}_k - \#\mathcal{T}_0 &\le M_p\sum_{j=0}^{k-1}E_k(u_k,\mathcal{T}_k)^{-1/s} + M_d\sum_{j=0}^{k-1}E_k(z_k,\mathcal{T}_k)^{-1/t}\\ &\le \left\{M_p\left(1+\frac{\gamma_p}{c_2}\right)^{1/2s}Q_k(u_k,\mathcal{T}_k)^{-1/s}\right.\\ &\qquad \left. +M_d\left(1+\frac{\gamma_d}{c_2}\right)^{1/2t}Q_k(z_k,\mathcal{T}_k)^{-1/t}\right\}\sum_{j=1}^{k}\alpha^{j/s}\end{aligned}$$

where the geometric series in $\alpha < 1$ is bounded by $S(\theta) = \alpha^{1/s}(1-\alpha^{1/s})^{-1}$. Then

$$\begin{aligned}\#\mathcal{T}_k - \#\mathcal{T}_0 \le S(\theta)&\left\{M_p\left(1+\frac{\gamma_p}{c_2}\right)^{1/2s}Q_k(u_k,\mathcal{T}_k)^{-1/s}\right.\\ &\left. +M_d\left(1+\frac{\gamma_d}{c_2}\right)^{1/2t}Q_k(u_k,\mathcal{T}_k)^{-1/t}\right\}\\ \le S(\theta)&\left\{M_p\left(1+\frac{\gamma_p}{c_2}\right)^{1/2s}\left(|\!|\!| u-u_k |\!|\!|^2+\gamma_p\mathrm{osc}_k^2(u_k,\mathcal{T}_k)\right)^{-1/2s}\right.\\ &\left. +\ M_d\left(1+\frac{\gamma_d}{c_2}\right)^{1/2t}\left(|\!|\!| z-u_k |\!|\!|^2+\gamma_d\mathrm{osc}_k^2(z_k,\mathcal{T}_k)\right)^{-1/2t}\right\}.\end{aligned}$$

As seen in (4.32) the total error and quasi-error are equivalent up to a constant so this result may be viewed with respect to either the quasi- or total-error. □

5. CONCLUSION

In this article we developed convergence theory for a class of goal-oriented adaptive finite element methods for second order nonsymmetric linear elliptic equations. In particular, we established contraction and quasi-optimality results for a method of this type for the elliptic problem (1.1)–(1.2) with A Lipschitz, almost-everywhere (a.e.) symmetric positive definite (SPD), with b divergence-free, and with $c \geq 0$. We first described the problem class in some detail, with a brief review of conforming finite element discretization and error-estimate-driven adaptive finite element methods (AFEM). We then described a goal-oriented variation of standard AFEM (GOAFEM). Following the recent work of Mommer and Stevenson [10] for symmetric problems, we established contraction of GOAFEM. We also showed convergence in the sense of the goal function. Our analysis approach was signficantly different from that of Mommer and Stevenson [10], and involved the combination of the recent contraction frameworks of Cascon et. al [4], Nochetto, Siebert, and Veeser [11], and of Holst, Tsogtgerel, and Zhu [8]. We also did a careful complexity analysis, and established quasi-optimal cardinality of GOAFEM.

Problems that were not yet addressed include allowing for jump discontinuities in the diffusion cofficient, and allowing for lower-order nonlinear terms. We will address both of these aspects in a future work.

6. APPENDIX

Duality. We include an appendix discussion of the duality argument used in the quasi-orthogonality estimate in an effort to make the paper more self-contained.

Let u the variational solution to (1.3) and $u_1 \in \mathbb{V}_1$ the Galerkin solution to (2.22). Assume for any $g \in L_2(\Omega)$ the solution w to the dual problem (2.1) belongs to $H^2(\Omega) \cap H^1_0(\Omega)$ and

$$|w|_{H^2(\Omega)} \leq K_R \|g\|_{L_2(\Omega)}. \tag{6.1}$$

Then

$$\|u - u_1\|_{L_2} \leq C h_0 |\!|\!| u - u_1 |\!|\!|. \tag{6.2}$$

If $w \in H^2_{\text{loc}}(\Omega) \cap H^1_0(\Omega)$ but $w \notin H^2(\Omega)$ due to the angles of a nonconvex polyhedral domain Ω then $w \in H^{1+s}$ for some $0 < s < 1$ where s depends on the angles of $\partial\Omega$. Assume in this case for any $g \in L_2$

$$|w|_{H^{1+s}(\Omega)} \leq K_R \|g\|_{L_2(\Omega)} \tag{6.3}$$

then

$$\|u - u_1\|_{L_2} \leq C h_0^s |\!|\!| u - u_1 |\!|\!|. \tag{6.4}$$

As discussed in [5], [6] and [1] the regularity assumptions are reasonable based on the continuity of the diffusion coefficients a_{ij} and the convection and reaction coefficients b_i and c in $L_\infty(\Omega)$.

Proof of (6.2)*:* The proof follows the duality arguments in [1] and [3].

Let $w \in H^1_0(\Omega)$ the solution to the dual problem

$$a^*(w, v) = \langle u - u_1, v \rangle, \quad v \in H^1_0(\Omega). \tag{6.5}$$

Let $\mathcal{I}^h$ a global interpolator based on refinement $\mathcal{T}_1$. Assume $\mathcal{I}^h w$ is C^0 and the corresponding shape functions have approximation order m. For $m = 2$

$$\|w - \mathcal{I}^h w\|_{H^1} \leq C_{\mathcal{I}} h_{\mathcal{T}_1} |w|_{H^2.} \tag{6.6}$$

As discussed in [1] the interpolation estimate over reference element $\hat{T}$ follows from the Bramble-Hilbert lemma applied to the bounded linear functional $f(\hat{u}) = \langle \hat{u} - \mathcal{I}^h \hat{u}, \hat{v} \rangle$

where $\hat{v} \in H^t(\hat{T})$ is arbitrary then set to $\hat{u} - \mathcal{I}^h \hat{u}$. The Sobolev semi-norms for $t = 0, 1$ over elements $T \in \mathcal{T}$ are bounded via change of variables to the reference element. Summing over $T \in \mathcal{T}$ and combining semi-norms into a norm estimate establishes (6.6).

By (6.1) we have the bound

$$|w|_{H^2} \leq K_R \|u - u_1\|_{L_2}. \tag{6.7}$$

By the identity $a(v, y) = a^*(y, v)$ write the primal form of the variational problems

$$a(u, v) = f(v), \quad v \in H_0^1(\Omega) \tag{6.8}$$

$$a(u_1, v) = f(v), \quad v \in \mathbb{V}_1 \tag{6.9}$$

$$a(v, w) = \langle u - u_1, v\rangle, \quad v \in H_0^1(\Omega). \tag{6.10}$$

Taking $v = u - u_1 \in H_0^1$ in (6.10)

$$a(u - u_1, w) = \langle u - u_1, u - u_1\rangle = \|u - u_1\|_{L_2}^2. \tag{6.11}$$

Combining (6.8) and (6.9) we have the Galerkin orthogonality result

$$a(u - u_1, v) = 0, \quad v \in \mathbb{V}_1. \tag{6.12}$$

Then by (6.11) and (6.12) noting the interpolant of the dual solution $\mathcal{I}^h w \in \mathbb{V}_1$

$$\|u - u_1\|_{L_2}^2 = a(u - u_1, w) = a(u - u_1, w - \mathcal{I}^h w). \tag{6.13}$$

Starting with (6.13) and applying continuity (2.8), interpolation estimate (6.6) and elliptic regularity (6.7)

$$\begin{aligned} \|u - u_1\|_{L_2}^2 &\leq M_c \|u - u_1\|_{H^1} \|w - \mathcal{I}^h w\|_{H^1} \\ &\leq M_c \|u - u_1\|_{H^1} C_{\mathcal{I}} h_{\mathcal{T}_1} |w|_{H^2} \\ &\leq K_R M_c C_{\mathcal{I}} h_0 \|u - u_1\|_{H^1} \|u - u_1\|_{L_2}. \end{aligned}$$

Canceling one factor of $\|u - u_1\|_{L_2}$ and applying coercivity (2.9)

$$\|u - u_1\|_{L_2} \leq \frac{M_c}{m_{\mathcal{E}}} C_{\mathcal{I}} K_R h_0 |\!|\!| u - u_1 |\!|\!|. \tag{6.14}$$

Depending on the regularity of the boundary $\partial\Omega$ the solution w may have less regularity: $w \in H^2_{\text{loc}(\Omega)}$ but $w \notin H^2(\Omega)$. In particular, we may have $w \in H^{1+s}$ for some $s \in (0, 1)$. In that case obtain the more general estimate

$$\|w - \mathcal{I}^h w\|_{H^1} \leq \tilde{C}_{\mathcal{I}} h_0^s |w|_{1+s}$$

yielding

$$\|u - u_1\|_{L_2} \leq \frac{M_c}{m_{\mathcal{E}}} \tilde{C}_{\mathcal{I}} K_R h_0^s |\!|\!| u - u_1 |\!|\!|.$$

The value of s is found by considering all corners of boundary $\partial\Omega$. Writing the interior angle at each corner by $\omega = \pi/\alpha$ it holds for $\alpha > 0$ and arbitrary $\varepsilon > 0$

$$\omega = \pi/\alpha \implies w \in H^{1+\alpha-\varepsilon}$$

and if $\pi/(p_j + 1) \leq \omega \leq \pi/p_j$ for a set of integers p_j characterizing the corners of $\partial\Omega$

$$\|w - \mathcal{I}^h w\|_{H^1} \leq C h^s |w|_{1+s}$$

where $s = \min\{p_j, 1\}$ and $s = 1$ in the case of a smooth boundary or a convex polyhedral domain. Details may be found in [1] and [13].

ACKNOWLEDGMENTS

MH was supported in part by NSF Awards 0715146 and 0915220 and DOD/DTRA Award HDTRA-09-1-0036. SP was supported in part by NSF Award 0715146.

REFERENCES

[1] O. Axelsson and V. A. Barker. *Finite element solution of boundary value problems: theory and computation*. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2001.

[2] P. Binev, W. Dahmen, and R. DeVore. Adaptive finite element methods with convergence rates. *Numer. Math.*, 97(2):219–268, 2004.

[3] S. Brenner and L. Scott. *The Mathematical Theory of Finite Element Methods*. Springer-Verlag, third edition, 2008.

[4] J. M. Cascon, C. Kreuzer, R. H. Nochetto, Kunibert, and G. Siebert. Quasi-optimal convergence rate for an adaptive finite element method, in preparation. *SIAM J. Numer. Anal.*, 46(5):2524–2550, 2008.

[5] P. G. Ciarlet. *Finite Element Method for Elliptic Problems*. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2002.

[6] L. C. Evans. *Partial Differential Equations (Graduate Studies in Mathematics, V. 19) GSM/19*. American Mathematical Society, 1998.

[7] T. Gilbarg. *Elliptic partial differential equations of second order*. Springer-Verlag, 1977.

[8] M. Holst, G. Tsogtgerel, and Y. Zhu. Local and global convergence of adaptive methods for nonlinear partial differential equations, 2008.

[9] K. Mekchay and R. Nochetto. Convergence of adaptive finite element methods for general second order linear elliptic PDE. *SINUM*, 43(5):1803–1827, 2005.

[10] M. S. Mommer and R. Stevenson. A goal-oriented adaptive finite element method with convergence rates. *SIAM J. Numer. Anal.*, 47(2):861–886, 2009.

[11] R. H. Nochetto, K. G. Siebert, and A. Veeser. *Theory of adaptive finite element methods: an introduction*, pages 409 – 542. Springer, 2009.

[12] R. Stevenson. The completion of locally refined simplicial partitions created by bisection. Technical report, 2006.

[13] G. Strang and G. J. Fix. *An Analysis of the Finite Element Method*. Prentice-Hall (Series in Automatic Computation), Englewood Clifs, N. J., 1973.

E-mail address: `mholst@math.ucsd.edu`

E-mail address: `snpolloc@math.ucsd.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA SAN DIEGO, LA JOLLA CA 92093