Far-from-constant mean curvature solutions of Einstein's constraint equations with positive Yamabe metrics

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We establish new existence results for the Einstein constraint equations for mean extrinsic curvature arbitrarily far from constant. The results hold for rescaled background metric in the positive Yamabe class, with freely specifiable parts of the data sufficiently small, and with matter energy density not identically zero. Two technical advances make these results possible: A new topological fixed-point argument without smallness conditions on spatial derivatives of the mean extrinsic curvature, and a new global supersolution construction for the Hamiltonian constraint that is similarly free of such conditions. The results are presented for strong solutions on closed manifolds, but also hold for weak solutions and for compact manifolds with boundary. These results are apparently the first that do not require smallness conditions on spatial derivatives of the mean extrinsic curvature.

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Introduction. The question of existence of solutions to the Lichnerowicz-York conformally rescaled Einstein's constraint equations, for an arbitrarily prescribed mean extrinsic curvature, has remained an open problem for more than thirty years [1]. The rescaled equations, which are a coupled nonlinear elliptic system consisting of the scalar Hamiltonian constraint coupled to the vector momentum constraint, have been studied almost exclusively in the setting of constant mean extrinsic curvature, known as the CMC case. In the CMC case the equations decouple, and it has long been known how to establish existence of solutions. The case of CMC data on closed (compact without boundary) manifolds was completely resolved by several authors over the last twenty years, with the last remaining subcases resolved and summarized by Isenberg in [2]. Over the last ten years, other CMC cases were studied and resolved; see the survey [3].

Conversely, the question of existence of solutions to the Einstein constraint equations for nonconstant mean extrinsic curvature (the "non-CMC case") has remained largely unanswered, with progress made only in the case that the mean extrinsic curvature is nearly constant (the "near-CMC case"), in the sense that the size of its spatial derivatives is sufficiently small. The near-CMC condition leaves the constraint equations coupled, but ensures the coupling is weak. In [4], Isenberg and Moncrief established the first existence (and uniqueness) result in the near-CMC case, for background metric having negative Ricci scalar. Their result was based on a fixed-point argument, together with the use of iteration barriers (sub- and supersolutions) which were shown to be bounded above and below by fixed positive constants, independent of the iteration. We note that both the fixed-point argument and the global barrier construction in [4] rely critically on the near-CMC assumption. All subsequent non-CMC existence results are based on the analysis framework in [4] and are thus limited to the near-CMC case (see the survey [3], the nonexistence results in [5], and also the newer existence results in [6] for non-negative Yamabe classes).

This article presents the first non-CMC existence results for the Einstein constraints that do not require the near-CMC assumption. Two recent advances make this possible: A new topological fixed-point argument (established in [7, 8]) and a new global supersolution construction for the Hamiltonian constraint (presented here and in [8]) that are both free of near-CMC conditions. These two results allow us to establish existence of non-CMC solutions for conformal background metrics in the positive Yamabe class, with the freely specifiable part of the data given by the traceless transverse part of the rescaled extrinsic curvature and the matter fields sufficiently small, and with the matter energy density not identically zero. We only state the main results and give the ideas of the proofs; detailed proofs may be found in [8] for closed manifolds and in [7] for compact manifolds with boundary. Our results here and in [7, 8] reduce the remaining open questions of existence of non-CMC solutions without near-CMC conditions to two basic open questions: (1) Existence of global *super*-solutions for background metrics in the nonpositive Yamabe classes and for large data; and (2) existence of global *sub*-solutions for background metrics in the positive Yamabe class in vacuum.

The Conformal Method. The manifold and fields $(\mathcal{M}, \hat{h}_{ab}, \hat{k}^{ab}, \hat{j}^a, \hat{\rho})$ form an *initial data set* for Einstein's equations iff $\mathcal{M}$ is a 3-dimensional smooth manifold, $\hat{h}_{ab}$ is a Riemannian metric on $\mathcal{M}$, $\hat{k}^{ab}$ is a symmetric tensor field on $\mathcal{M}$, $\hat{j}^a$ and $\hat{\rho}$ are a vector field and a non-negative scalar field on $\mathcal{M}$, respectively, satisfying an energy condition (described below), and the following hold on $\mathcal{M}$,

$$\hat{R} + \hat{k}^2 - \hat{k}_{ab}\hat{k}^{ab} - 2\kappa\hat{\rho} = 0, \quad (1)$$

$$-\hat{\nabla}_a\hat{k}^{ab} + \hat{\nabla}^b\hat{k} + \kappa\hat{j}^b = 0. \quad (2)$$

Here, $\hat{\nabla}_a$ is the Levi-Civita connection of $\hat{h}_{ab}$, so it satisfies $\hat{\nabla}_a\hat{h}_{bc} = 0$, $\hat{R}$ is the Ricci scalar of the connection $\hat{\nabla}_a$,

$\hat{k} = \hat{h}_{ab}\hat{k}^{ab}$ is the trace of $\hat{k}^{ab}$, and $\kappa = 8\pi$ in units where both the gravitational constant and speed of light have value one. We denote by $\hat{h}^{ab}$ the tensor inverse of $\hat{h}_{ab}$. Tensor indices of hatted quantities are raised and lowered with $\hat{h}^{ab}$ and $\hat{h}_{ab}$, respectively. When (1)-(2) hold, the manifold $\mathcal{M}$ can be embedded as a hyper-surface in a 4-dimensional manifold corresponding to a solution of the space-time Einstein field equations, and the push-forward of $\hat{h}^{ab}$ and $\hat{k}^{ab}$ represent the first and second fundamental forms of the embedded hyper-surface. This leads to the terminology extrinsic curvature for $\hat{k}^{ab}$, and mean extrinsic curvature for its trace, $\hat{k}$. The dominant energy condition on the matter fields implies the energy condition $-\hat{\rho}^2 + \hat{h}_{ab}\hat{j}^a\hat{j}^b \leqslant 0$, with strict inequality at points on $\mathcal{M}$ where $\rho \neq 0$; see [9]. This condition is why the trivial procedure of fixing an arbitrary Riemannian metric $\hat{h}_{ab}$ and a symmetric tensor $\hat{k}^{ab}$ and then defining $\hat{j}^a$ and $\hat{\rho}$ by (1)-(2) does not generally give a physically meaningful initial data set for Einstein's equations.

The conformal method consists of finding solutions $\hat{h}_{ab}$, $\hat{k}^{ab}$, $\hat{j}^a$ and $\hat{\rho}$ of (1)-(2) using a particular decomposition. To proceed, fix on $\mathcal{M}$ a Riemannian metric h_{ab} with Levi-Civita connection ∇_a, so it satisfies $\nabla_a h_{bc} = 0$, and has Ricci scalar R. Fix on $\mathcal{M}$ a symmetric tensor σ^{ab}, trace-free and divergence-free with respect to h_{ab}, that is, $h_{ab}\sigma^{ab} = 0$ and $\nabla_a\sigma^{ab} = 0$. Also fix on $\mathcal{M}$ scalar fields τ and ρ, and a vector field j^a, subject to the condition $-\rho^2 + h_{ab}j^aj^b \leqslant 0$, with strict inequality at points on $\mathcal{M}$ where $\rho \neq 0$. We have denoted by h^{ab} the tensor inverse of h_{ab}, and we use the convention that tensor indices of unhatted quantities are raised and lowered with the tensors h^{ab} and h_{ab}, respectively. Finally, given a smooth vector field w^a on $\mathcal{M}$, introduce the conformal Killing operator $\mathcal{L}$ as follows, $(\mathcal{L}w)^{ab} = \nabla^a w^b + \nabla^b w^a - (2/3)\,(\nabla_c w^c)\,h^{ab}$. The conformal method then involves first solving the following equations for a scalar field ϕ and vector field w^a

$$-\Delta\phi + a_R\phi + a_\tau\,\phi^5 - a_w\,\phi^{-7} - a_\rho\,\phi^{-3} = 0, \qquad (3)$$
$$-\nabla_a(\mathcal{L}w)^{ab} + b^b_\tau\phi^6 + b^b_j = 0, \qquad (4)$$

where we have introduced the Laplace-Beltrami operator $\Delta\phi = h^{ab}\nabla_a\nabla_b\phi$, and the functions $a_R = R/8$, $a_\tau = \tau^2/12$, $a_\rho = \kappa\rho/4$, $b^b_\tau = (2/3)\nabla^b\tau$, $b^b_j = \kappa j^b$, and $a_w = \big[\sigma_{ab} + (\mathcal{L}w)_{ab}\big]\big[\sigma^{ab} + (\mathcal{L}w)^{ab}\big]/8$. One then recovers the tensors $\hat{h}_{ab}$, $\hat{k}^{ab}$, $\hat{j}^a$ and $\hat{\rho}$ through the expressions

$$\hat{h}_{ab} = \phi^4 h_{ab}, \quad \hat{j}^a = \phi^{-10}\,j^a, \quad \hat{\rho} = \phi^{-8}\,\rho, \qquad (5)$$
$$\hat{k}^{ab} = \phi^{-10}\big[\sigma^{ab} + (\mathcal{L}w)^{ab}\big] + \frac{1}{3}\phi^{-4}\tau\,h^{ab}. \qquad (6)$$

A straightforward computation shows that if $\hat{h}_{ab}$, $\hat{k}^{ab}$, $\hat{j}^a$ and $\hat{\rho}$ have the form given in (5)-(6), then equations (1)-(2) are equivalent to (3)-(4). Hatted fields represent quantities with physical meaning, except the trace τ of the physical extrinsic curvature $\hat{k}^{ab}$, that is, $\tau = \hat{k}$.

We employ standard L^p and Sobolev spaces $W^{k,p}$, following [10] for scalar-valued functions on bounded sets in $\mathbb{R}^n$, and following [11] and [12] for generalizations to manifolds and to tensor fields. The space L^∞ is the set of almost everywhere (a.e.) bounded functions on $\mathcal{M}$, which is a Banach space with norm $\|u\|_\infty :=$ ess $\sup_{\mathcal{M}} |u|$. The Banach space L^p, with $1 \leqslant p < \infty$, is the set of tensor fields on $\mathcal{M}$ having norm $\|u\|_p := \left[\int_{\mathcal{M}} (u_{a_1\cdots a_n} u^{a_1\cdots a_n})^{p/2} dx\right]^{1/p}$ finite. The Banach space $W^{k,p}$ is the set of tensor fields on $\mathcal{M}$ having $k \geqslant 0$ weak covariant derivatives in L^p, with norm denoted $\|\ \|_{k,p}$.

The Momentum Constraint. The momentum constraint (4) is well-understood in the case that h_{ab} has no conformal Killing vectors (a vector field v^a is *conformal Killing* iff $(\mathcal{L}v)^{ab} = 0$). A standard result is the following. Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional, closed, C^2, Riemannian manifold, with h_{ab} having no conformal Killing vectors, and let b^a_τ, $b^a_j \in L^p$ with $p \geqslant 2$ and $\phi \in L^\infty$; Then, equation (4) has a unique solution $w^a \in W^{2,p}$ with

$$c\,\|w\|_{2,p} \leqslant \|\phi\|^6_\infty\,\|b_\tau\|_p + \|b_j\|_p, \qquad (7)$$

where $c > 0$ is a constant. We have generalized this result in [7, 8], allowing weaker coefficient differentiability, giving existence of solutions down to $w^a \in W^{1,p}$, with real number $p \geqslant 2$. The proof in [8] is based on Riesz-Schauder theory for compact operators [13]. The case of compact manifold $\mathcal{M}$ with boundary is analyzed in [7].

From inequality (7) it is not difficult to show that for $p > 3$ the following pointwise estimate holds,

$$a_w \leqslant K_1\,\|\phi\|^{12}_\infty + K_2, \qquad (8)$$

with $K_1 = \frac{1}{2}(\frac{c_s c_{\mathcal{L}}}{c})^2\|b_\tau\|^2_p$, $K_2 = \frac{1}{4}\|\sigma\|^2_\infty + \frac{1}{2}(\frac{c_s c_{\mathcal{L}}}{c})^2\|b_j\|^2_p$, where c_s is the constant in the embedding $W^{1,p} \hookrightarrow L^\infty$, and $c_{\mathcal{L}}$ is a bound on the norm of $\mathcal{L} : W^{2,p} \to W^{1,p}$. There is no smallness assumption on $\|b_\tau\|_p$, so the near-CMC condition is not required for these results.

Global Hamiltonian Constraint Barriers. Let $\mathcal{M}$ be closed. The scalar functions ϕ_- and ϕ_+ are called *barriers* (sub- and supersolutions, respectively) iff

$$-\Delta\phi_- + a_R\phi_- + a_\tau\,\phi^5_- - a_w\,\phi^{-7}_- - a_\rho\,\phi^{-3}_- \leqslant 0, \qquad (9)$$
$$-\Delta\phi_+ + a_R\phi_+ + a_\tau\,\phi^5_+ - a_w\,\phi^{-7}_+ - a_\rho\,\phi^{-3}_+ \geqslant 0. \qquad (10)$$

The barriers are *compatible* iff $0 < \phi_- \leqslant \phi_+$, and are *global* iff (9)–(10) holds for all w^a solving equation (4), with source $\phi \in [\phi_-, \phi_+]$. The closed interval

$$[\phi_-, \phi_+] = \{\phi \in L^p : \phi_- \leqslant \phi \leqslant \phi_+ \text{ a.e. in } \mathcal{M}\}, \qquad (11)$$

is a topologically closed subset of L^p, $1 \leqslant p \leqslant \infty$ (see [8]). Global supersolutions are difficult to find as a consequence of the non-negativity of a_w and its estimate (8), together with the limit (11). All previous global supersolution constructions, such as those in [4, 6], rely in a critical way on the near-CMC assumption, which appears as the condition that a suitable norm of $\nabla\tau$ be sufficiently small, or equivalently, that K_1 in (8) be sufficiently small. The main result in this letter is to establish existence of global supersolutions of the Hamiltonian constraint without the near-CMC assumption. We need the following notation: Given any scalar function $v \in L^\infty$, denote by $v^\wedge = \text{ess sup}_{\mathcal{M}} v$, and $v^\vee = \text{ess inf}_{\mathcal{M}} v$.

Theorem 1 *Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional, smooth, closed Riemannian manifold with metric h_{ab} in the positive Yamabe class with no conformal Killing vectors. Let u be a smooth positive solution of the Yamabe problem*

$$-\Delta u + a_R u - u^5 = 0, \tag{12}$$

and define the constant $k = u^\wedge/u^\vee$. If the function τ is nonconstant and the rescaled matter fields j^a, ρ, and traceless transverse tensor σ^{ab} are sufficiently small, then

$$\phi_+ = \epsilon u, \quad \epsilon = \left[\frac{1}{2K_1 k^{12}}\right]^{\frac{1}{4}} \tag{13}$$

is a global supersolution of equation (3).

Proof. (Theorem 1) Existence of a smooth positive solution u to (12) is summarized in [14]. Using the notation

$$E(\phi) = -\Delta\phi + a_R\phi + a_\tau\phi^5 - a_w\phi^{-7} - a_\rho\phi^{-3}, \tag{14}$$

we have to show $E(\phi_+) \geqslant 0$. Taking $\phi_+ = \epsilon u$, $\epsilon > 0$ gives the identity $-\Delta\phi_+ + a_R\phi_+ = \epsilon u^5$. We have

$$\begin{aligned} E(\phi_+) &\geqslant -\Delta\phi_+ + a_R\phi_+ - \frac{K_1(\phi_+^\wedge)^{12} + K_2}{\phi_+^7} - \frac{a_\rho^\wedge}{\phi_+^3} \\ &\geqslant \epsilon\, u^5 - K_1\Big[\frac{\phi_+^\wedge}{\phi_+^\vee}\Big]^{12}\phi_+^5 - \frac{K_2}{\phi_+^7} - \frac{a_\rho^\wedge}{\phi_+^3} \\ &\geqslant \epsilon u^5\Big[1 - K_1\, k^{12}\epsilon^4 - \frac{K_2}{\epsilon^8 u^{12}} - \frac{a_\rho^\wedge}{\epsilon^4 u^8}\Big], \end{aligned}$$

where we have used $\phi_+^\wedge/\phi_+^\vee = u^\wedge/u^\vee = k$. The choice of ϵ made in (13) is equivalent to $1/2 = 1 - K_1\, k^{12}\epsilon^4$. For this ϵ, impose on the free data σ^{ab}, ρ and j^a the condition

$$\frac{1}{2} - \frac{K_2}{\epsilon^8(u^\vee)^{12}} - \frac{a_\rho^\wedge}{\epsilon^4(u^\vee)^8} \geqslant 0.$$

Thus for any $K_1 > 0$, we can guarantee $E(\phi_+) \geqslant 0$ for sufficiently small σ^{ab}, ρ and j^a. □

Theorem 1 shows that global supersolutions ϕ_+ can be built without using near-CMC conditions by rescaling solutions to the Yamabe problem (12); the larger $\|\nabla\tau\|_p$, the smaller the factor ϵ. Existence of the finite positive constant k appearing in Theorem 1 is related to establishing a Harnack inequality for solutions to the Yamabe problem (see [15]). It remains to construct (again, without near-CMC conditions) a compatible global subsolution satisfying $0 < \phi_- \leqslant \phi_+$. We now give a variant of some known constructions [16, 17, 18]; so also [7, 8].

Theorem 2 *Let the assumptions for Theorem 1 hold. If also the rescaled matter energy density ρ is not identically zero, then there exists a positive global subsolution ϕ_- of equation (3), compatible with the global supersolution in Theorem 1, so that it satisfies $0 < \phi_- \leqslant \phi_+$.*

Proof. (Theorem 2) Let $a_\rho \geqslant \zeta > 0$ in some open set $B \subset \mathcal{M}$. We know from [2] that there exists u satisfying

$$-\Delta u + a_R u - R_u u^5 = 0, \tag{15}$$

such that $R_u \leqslant -\xi < 0$ in $\mathcal{M} \setminus B$. Taking $\phi_- = \epsilon u$, $\epsilon > 0$ gives the identity $-\Delta\phi_- + a_R\phi_- = \epsilon R_u u^5$. Using $E(\phi)$ from (14), we must show $E(\phi_-) \leqslant 0$. We have

$$\begin{aligned} E(\phi_-) &= -\Delta\phi_- + a_R\phi_- + a_\tau\phi_-^5 - a_w\phi_-^{-7} - a_\rho\phi_-^{-3} \\ &\leqslant \epsilon R_u(u^\vee)^5 + a_\tau^\wedge\epsilon^5(u^\wedge)^5 - a_\rho\epsilon^{-3}(u^\vee)^{-3}. \end{aligned}$$

Now find $\epsilon = \epsilon_1 > 0$ sufficiently small so on $B \subset \mathcal{M}$,

$$\epsilon_1 R_u(u^\vee)^5 + a_\tau^\wedge\epsilon_1^5(u^\wedge)^5 - \zeta\epsilon_1^{-3}(u^\vee)^{-3} \leqslant 0.$$

Next find $\epsilon = \epsilon_2 > 0$ sufficiently small so on $\mathcal{M} \setminus B$,

$$-\xi\epsilon_2(u^\vee)^5 + a_\tau^\wedge\epsilon_2^5(u^\wedge)^5 \leqslant 0.$$

Taking now $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\} > 0$ produces a global subsolution $\phi_- = \epsilon u$, for any $\epsilon \in (0, \epsilon_0]$. We now finally take $\epsilon \in (0, \epsilon_0]$ sufficiently small so that $0 < \phi_- \leqslant \phi_+$. □

The Hamiltonian Constraint. We now state some supporting results we need from [7, 8] for solutions of (3). We state only the results for strong solutions, recovering previous results in [2, 4]. Generalizations allowing weaker differentiability conditions on the coefficients appear in [7, 8].

Theorem 3 *Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional, C^2, closed Riemannian manifold. Let the free data τ^2, σ^2 and ρ be in L^p, with $p \geqslant 2$. Let ϕ_- and ϕ_+ be barriers to (3) for a particular value of the vector $w^a \in W^{1,2p}$. Then, there exists a solution $\phi \in [\phi_-, \phi_+] \cap W^{2,p}$ of the Hamiltonian constraint (3). Furthermore, if the metric h_{ab} is in the non-negative Yamabe classes, then ϕ is unique.*

Proof. (Theorem 3) The proofs in [7, 8] make use of barriers, a priori estimates, and variational methods. □

The Coupled Constraint System. Our main result concerning the coupled constraint system is the following.

Theorem 4 *Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional, smooth, closed Riemannian manifold with metric h_{ab} in the positive Yamabe class with no conformal Killing vectors. Let $p > 3$ and let τ be in $W^{1,p}$. Let σ^2, j^a, and ρ be in L^p and satisfy the assumptions for Theorems 1 and 2 to yield a compatible pair of global barriers $0 < \phi_- \leqslant \phi_+$ to the Hamiltonian constraint (3). Then, there exists a scalar field $\phi \in [\phi_-, \phi_+] \cap W^{2,p}$ and a vector field $w^a \in W^{2,p}$ solving the constraint equations (3)-(4).*

Theorem 4 can be proven using the following topological fixed-point result established in [7, 8]. For a review of reflexive and ordered Banach spaces, see [7, 8, 19]. Note that such compactness arguments do not give uniqueness.

Lemma 1 *Let X and Y be Banach spaces, and let Z be a Banach space with compact embedding $X \hookrightarrow Z$. Let $U \subset Z$ be a nonempty, convex, bounded subset which is closed in the topology of Z, and let the maps*

$$S : U \to \mathcal{R}(S) \subset Y, \qquad T : U \times \mathcal{R}(S) \to U \cap X,$$

be continuous. Then there exist $w \in \mathcal{R}(S)$ and $\phi \in U \cap X$ such that

$$\phi = T(\phi, w) \quad \textit{and} \quad w = S(\phi). \tag{16}$$

Proof. (Lemma 1) The proofs of this result and several useful variations appear in [7, 8]. □

Proof. (Theorem 4) The proof is through Lemma 1. First, for arbitrary real number $s > 0$, express (3)-(4) as

$$L_s\phi + f_s(\phi, w) = 0, \quad (\boldsymbol{L}w)^a + \boldsymbol{f}(\phi)^a = 0, \tag{17}$$

where $L_s : W^{2,p} \to L^p$ and $\boldsymbol{L} : W^{2,p} \to L^p$ are defined as $L_s\phi := [-\Delta + s]\phi$, and $(\boldsymbol{L}w)^a := -\nabla_b(\mathcal{L}w)^{ab}$, and where $f_s : [\phi_-, \phi_+] \times W^{2,p} \to L^p$ and $\boldsymbol{f} : [\phi_-, \phi_+] \to L^p$ are

$$f_s(\phi, w) := [a_R - s]\phi + a_\tau\phi^5 - a_w\phi^{-7} - a_\rho\phi^{-3},$$
$$\boldsymbol{f}(\phi)^a := b^a_\tau\phi^6 + b^a_j.$$

Introduce now the operators $S : [\phi_-, \phi_+] \to W^{2,p}$ and $T : [\phi_-, \phi_+] \times W^{2,p} \to W^{2,p}$ which are given by

$$S(\phi)^a := -[\boldsymbol{L}^{-1}\boldsymbol{f}(\phi)]^a, \quad T(\phi, w) := -L_s^{-1} f_s(\phi, w).$$

The mappings S and T are well-defined due to the absence of conformal Killing vectors and by introduction of the positive shift $s > 0$, ensuring both $\boldsymbol{L}$ and L_s are invertible (see [7, 8]). The equations (17) have the form (16) for use of Lemma 1. We have the Banach spaces $X = W^{2,p}$ and $Y = W^{2,p}$, and the (ordered) Banach space $Z = L^\infty$ with compact embedding $W^{2,p} \hookrightarrow L^\infty$. The compatible barriers form the nonempty, convex, bounded L^∞-interval $U = [\phi_-, \phi_+]$, which we noted earlier is closed in L^p for $1 \leqslant p \leqslant \infty$ (see [8]). It remains to show S and T are continuous maps. These properties follow from equation (7) and from Theorem 3 with global barriers from Theorem 1 and Theorem 2, using standard inequalities. Theorem 4 now follows from Lemma 1. □

See [8] for generalizations of Theorem 4 to arbitrary space dimensions and allowing weaker differentiability conditions on the coefficients, establishing existence of nonvacuum, non-CMC weak solutions down to $\phi \in W^{s,p}$, for $(s-1)p > 3$. Generalizations of the results here and in [8] to compact manifolds with boundary appear in [7].

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