ON CERTAIN GEOMETRIC OPERATORS BETWEEN SOBOLEV SPACES OF SECTIONS OF TENSOR BUNDLES ON COMPACT MANIFOLDS EQUIPPED WITH ROUGH METRICS

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ABSTRACT. The study of Einstein constraint equations in general relativity naturally leads to considering Riemannian manifolds equipped with nonsmooth metrics. There are several important differential operators on Riemannian manifolds whose definitions depend on the metric: gradient, divergence, Laplacian, covariant derivative, conformal Killing operator, and vector Laplacian, among others. In this article, we study the approximation of such operators, defined using a rough metric, by the corresponding operators defined using a smooth metric. This paves the road to understanding to what extent the nice properties such operators possess, when defined with smooth metric, will transfer over to the corresponding operators defined using a nonsmooth metric. These properties are often assumed to hold when working with rough metrics, but to date the supporting literature is slim.

CONTENTS

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1. INTRODUCTION

The study of Einstein constraint equations in general relativity naturally leads to considering Riemannian manifolds equipped with metrics that are not C^∞ (see e.g. [9, 10, 18, 14, 3]). Some of the motivation for developing this understanding came from studies of the Einstein evolution equation with rough metric [15, 16]. In order to fully understand the implications of a rough metric, one needs to understand the impact of a nonsmooth metric on the various geometric and differential operators that arise in the formulation of stationary and evolution problems on Riemannian manifolds. The questions we study in this article fall into the following general form: Let (M^n, g) be a compact Riemannian manifold. Suppose $g \in W^{s,p}(T^2M)$ where $sp > n$ (it is reasonable to assume that the metric is continuous; the condition $sp > n$ guarantees that g has a continuous representative, and also it implies that $W^{s,p}(M)$ is a Banach algebra, which plays an important role in some of the calculations). Let $\{g_m\}$ be a sequence of smooth Riemannian metrics on M such that $g_m \to g$ in $W^{s,p}(T^2M)$. For each m, let A_m be an operator whose definition depends on the metric g_m. Let A be the corresponding operator that is defined in terms of g. What can be said about the relationship between the operators that are defined in terms of g_m and those that are defined in terms of g? Does $\{A_m\}$ converge to A (in an appropriate norm)? In particular, we are interested in the gradient, Laplacian, divergence, covariant derivative, and vector Laplacian operators. Additionally, we will study the relationship between the corresponding Riemannian curvature tensors, Ricci curvatures, and scalar curvatures.

One of the main applications of such results is in the study of elliptic partial differential equations on manifolds. An example of the type of question we hope to address is the following: the Laplacian and vector Laplacian of a smooth metric on a compact Riemannian manifold are Fredholm of index zero. Considering that the index of an operator is locally constant, in order to see whether this useful property carries over to the case of nonsmooth metrics we need to determine whether the Laplacian or vector Laplacian defined using a nonsmooth metric can be approximated by corresponding operators defined by smooth metrics. Results of this type and other related results have been used in literature without complete proof; they are well-motivated and reasonable assumptions in most cases, but it seems that a careful study is missing in the literature. This is particularly true in the case of noninteger Sobolev classes. In this manuscript, we have attempted to fill some of the gaps. This paper can be viewed as a part of our efforts to build a more complete foundation for the study of differential operators and Sobolev-Slobodeckij spaces on manifolds through a sequence of related manuscripts [4, 7, 6].

Outline of Paper. In Section 2 we summarize some of the basic definitions, notation and conventions used throughout the paper. In Section 3 we go over some backround material on analysis and differential geometry. In sections 4-14 we rigorously study the aforementioned question of convergence for various geometric operators that appear in the study of elliptic partial differential equations on compact manifolds.

2. NOTATION AND CONVENTIONS

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{N}$ denotes the set of positive integers, and $\mathbb{N}_0$ denotes the set of nonnegative integers. For any nonnegative real number s, the integer part of s is denoted by $\lfloor s \rfloor$. The letter n is a positive integer and stands for the dimension of the space. For all $k \in \mathbb{N}$, GL$(k, \mathbb{R})$ is the set of all $k \times k$ invertible matrices with real entries.

Ω is a nonempty open set in $\mathbb{R}^n$. The collection of all compact subsets of Ω will be denoted by $\mathcal{K}(\Omega)$. Lipschitz domain in $\mathbb{R}^n$ refers to a nonempty bounded open set in $\mathbb{R}^n$ with Lipschitz continuous boundary.

Each element of $\mathbb{N}_0^n$ is called a multi-index. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$, we let $|\alpha| := \alpha_1 + \cdots + \alpha_n$. Also, for sufficiently smooth functions $u : \Omega \to \mathbb{R}$ (or for any distribution u) we define the αth order partial derivative of u as follows:

$$\partial^\alpha u := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\,.$$

We use the notation $A \preceq B$ to mean $A \leq cB$, where c is a positive constant that does not depend on the non-fixed parameters appearing in A and B. We write $A \simeq B$ if $A \preceq B$ and $B \preceq A$.

We write $L(X, Y)$ for the space of all *continuous* linear maps from the normed space X to the normed space Y. We use the notation $X \hookrightarrow Y$ to mean $X \subseteq Y$ and the inclusion map is continuous.

Definition 2.1. *Let Ω be a nonempty open set in $\mathbb{R}^n$ and $m \in \mathbb{N}_0$.*

$C(\Omega) = \{f : \Omega \to \mathbb{R} : f \text{ is continuous}\}$

$C^m(\Omega) = \{f : \Omega \to \mathbb{R} : \forall\, |\alpha| \leq m \quad \partial^\alpha f \in C(\Omega)\} \qquad (C^0(\Omega) = C(\Omega))$

$BC(\Omega) = \{f : \Omega \to \mathbb{R} : f \text{ is continuous and bounded on } \Omega\}$

$BC^m(\Omega) = \{f \in C^m(\Omega) : \forall\, |\alpha| \leq m \quad \partial^\alpha f \text{ is bounded on } \Omega\}$

$BC(\bar{\Omega}) = \{f : \Omega \to \mathbb{R} : f \in BC(\Omega) \text{ and } f \text{ is uniformly continuous on } \Omega\}$

$BC^m(\bar{\Omega}) = \{f : \Omega \to \mathbb{R} : f \in BC^m(\Omega), \forall\, |\alpha| \leq m \quad \partial^\alpha f \text{ is uniformly continuous on } \Omega\,\}$

$C^\infty(\Omega) = \bigcap_{m\in\mathbb{N}_0} C^m(\Omega), \quad BC^\infty(\Omega) = \bigcap_{m\in\mathbb{N}_0} BC^m(\Omega), \quad BC^\infty(\bar{\Omega}) = \bigcap_{m\in\mathbb{N}_0} BC^m(\bar{\Omega})$

$C_c^\infty(\Omega) = \{f \in C^\infty(\Omega) : \text{support of } f \text{ is an element of } \mathcal{K}(\Omega)\}$

Remark 2.2. [1] *If $f : \Omega \to \mathbb{R}$ is in $BC(\bar{\Omega})$, then it possesses a unique, bounded, continuous extension to the closure $\bar{\Omega}$ of Ω.*

Definition 2.3. *Let Ω be a nonempty open set in $\mathbb{R}^n$. Let $s \in \mathbb{R}$ and $p \in (1, \infty)$.*

- *If $s = k \in \mathbb{N}_0$,*

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : \|u\|_{W^{k,p}(\Omega)} := \sum_{|\nu|\leq k} \|\partial^\nu u\|_{L^p(\Omega)} < \infty\}$$

- *If $s = \theta \in (0, 1)$,*

$$W^{\theta,p}(\Omega) = \{u \in L^p(\Omega) : |u|_{W^{\theta,p}(\Omega)} := \Big(\int\int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+\theta p}} dxdy\Big)^{\frac{1}{p}} < \infty\}$$

- *If $s = k + \theta$, $k \in \mathbb{N}_0$, $\theta \in (0, 1)$,*

$$W^{s,p}(\Omega) = \{u \in W^{k,p}(\Omega) : \|u\|_{W^{s,p}(\Omega)} := \|u\|_{W^{k,p}(\Omega)} + \sum_{|\nu|=k} |\partial^\nu u|_{W^{\theta,p}(\Omega)} < \infty\}$$

- *$W_0^{s,p}(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ in $W^{s,p}(\Omega)$.*

- *If* $s<0$,
$$W^{s,p}(\Omega)=(W_0^{-s,p'}(\Omega))^* \quad (\frac{1}{p}+\frac{1}{p'}=1)$$
- *For all compact sets* $K\subset\Omega$ *we define*
$$W_K^{s,p}(\Omega)=\{u\in W^{s,p}(\Omega): \textit{supp}\, u\subseteq K\}$$
with $\|u\|_{W_K^{s,p}(\Omega)}:=\|u\|_{W^{s,p}(\Omega)}$.
- $W_{loc}^{s,p}(\Omega):=\{u\in D'(\Omega):\forall\varphi\in C_c^\infty(\Omega)\quad \varphi u\in W^{s,p}(\Omega)\}$ *where* $D'(\Omega)$ *is the space of distributions on* Ω. $W_{loc}^{s,p}(\Omega)$ *is equipped with the natural topology induced by the separating family of seminorms* $\{|.|_\varphi\}_{\varphi\in C_c^\infty(\Omega)}\}$ *where*
$$\forall\, u\in W_{loc}^{s,p}(\Omega)\quad \varphi\in C_c^\infty(\Omega)\qquad |u|_\varphi:=\|\varphi u\|_{W^{s,p}(\Omega)}\,.$$

Let X, Y, and Z be Sobolev spaces (or locally Sobolev spaces). In this manuscript, by writing
$$X\times Y\hookrightarrow Z$$
we mean that the product of an element of X with an element of Y is an element of Z and moreover this multiplication is continuous in the following sense: if $u_i\to u$ in X and $v_i\to v$ in Y, then $u_i v_i\to uv$ in Z.

Throughout this manuscript, all manifolds are assumed to be smooth, Hausdorff, and second countable. We usually use the letter M for manifolds. If M is an n-dimensional smooth manifold, sometimes we use the shorthand notation M^n to indicate that M is n-dimensional.

Definition 2.4.

- *We say that a smooth atlas for a smooth manifold* M *is a* ***geometrically Lipschitz (GL)*** *smooth atlas if the image of each coordinate domain in the atlas under the corresponding coordinate map is a nonempty bounded open set with Lipschitz boundary.*
- *We say that a smooth atlas for a smooth manifold* M^n *is a* ***generalized geometrically Lipschitz (GGL)*** *smooth atlas if the image of each coordinate domain in the atlas under the corresponding coordinate map is the entire* $\mathbb{R}^n$ *or a nonempty bounded open set with Lipschitz boundary.*
- *We say that a smooth atlas for a smooth manifold* M^n *is a* ***nice*** *smooth atlas if the image of each coordinate domain in the atlas under the corresponding coordinate map is a ball in* $\mathbb{R}^n$.
- *We say that a smooth atlas for a smooth manifold* M^n *is a* ***super nice*** *smooth atlas if the image of each coordinate domain in the atlas under the corresponding coordinate map is the entire* $\mathbb{R}^n$.
- *We say that two smooth atlases* $\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in I}$ *and* $\{(\tilde{U}_\beta,\tilde{\varphi}_\beta)\}_{\beta\in J}$ *for a smooth manifold* M^n *are* ***geometrically Lipschitz compatible (GLC)*** *smooth atlases provided that each atlas is GGL and moreover for all* $\alpha\in I$ *and* $\beta\in J$ *with* $U_\alpha\cap\tilde{U}_\beta\neq\emptyset$, $\varphi_\alpha(U_\alpha\cap\tilde{U}_\beta)$ *and* $\tilde{\varphi}_\beta(U_\alpha\cap\tilde{U}_\beta)$ *are nonempty bounded open sets with Lipschitz boundary or the entire* $\mathbb{R}^n$.

Clearly, every super nice smooth atlas is also a GGL smooth atlas; every nice smooth atlas is also a GL smooth atlas, and every GL smooth atlas is also a GGL smooth atlas. Also, note that two arbitrary GL smooth atlases are not necessarily GLC smooth atlases because the intersection of two Lipschitz domains is not necessarily Lipschitz (see e.g.

[2], pages 115-117).

The tangent space of a manifold M^n at point $p \in M$ is denoted by T_pM, and the cotangent space by T_p^*M. If $(U, \varphi = (x^i))$ is a local coordinate chart and $p \in U$, we denote the corresponding coordinate basis for T_pM by $\partial_i|_p$ while $\frac{\partial}{\partial x^i}|_x$ denotes the basis for the tangent space to $\mathbb{R}^n$ at $x = \varphi(p) \in \mathbb{R}^n$; that is,

$$\varphi_* \partial_i = \frac{\partial}{\partial x^i}\,.$$

Note that for any smooth function $f : M \to \mathbb{R}$ we have

$$(\partial_i f) \circ \varphi^{-1} = \frac{\partial}{\partial x^i}(f \circ \varphi^{-1})\,.$$

The vector space of all k-covariant, l-contravariant tensors on T_pM is denoted by $T_l^k(T_pM)$. So each element of $T_l^k(T_pM)$ is a multilinear map of the form

$$F : \underbrace{T_p^*M \times \cdots \times T_p^*M}_{l \text{ copies}} \times \underbrace{T_pM \times \cdots \times T_pM}_{k \text{ copies}} \to \mathbb{R}\,.$$

Let M be a smooth manifold. A *(smooth real) vector bundle* of rank r over M is a smooth manifold E together with a surjective smooth map $\pi : E \to M$ such that

(1) for each $x \in M$, $E_x = \pi^{-1}(x)$ is an r-dimensional (real) vector space.
(2) for each $x \in M$, there exists a neighborhood U of x in M and a smooth map $\rho = (\rho^1, \cdots, \rho^r)$ from $E|_U := \pi^{-1}(U)$ onto $\mathbb{R}^r$ such that
   - for every $x \in U$, $\rho|_{E_x} : E_x \to \mathbb{R}^r$ is an isomorphism of vector spaces
   - $\Phi = (\pi|_{E_U}, \rho) : E_U \to U \times \mathbb{R}^r$ is a diffeomorphism.

The expressions "E is a vector bundle over M", or "$E \to M$ is a vector bundle", or "$\pi : E \to M$ is a vector bundle" are all considered to be equivalent. The space E is called the *total space* of the vector bundle $E \to M$. For each $x \in M$, $E_x := \pi^{-1}(x)$ is called the fiber over x. We refer to both $\Phi : E_U \to U \times \mathbb{R}^r$ and $\rho : E_U \to \mathbb{R}^r$ as a (smooth) *local trivialization* of E over U. We say that $E|_U$ is trivial. The pair (U, ρ) (or (U, Φ)) is sometimes called a *vector bundle chart*. It is easy to see that if (U, ρ) is a vector bundle chart and $\emptyset \neq V \subseteq U$ is open, then $(V, \rho|_{E_V})$ is also a vector bundle chart for E. Moreover if V is any nonempty open subset of M, then E_V is a vector bundle over the manifold V. We say that a triple (U, φ, ρ) is a **total trivialization triple** of the vector bundle $\pi : E \to M$ provided that (U, φ) is a smooth coordinate chart and $\rho = (\rho^1, \cdots, \rho^r) : E_U \to \mathbb{R}^r$ is a trivialization of E over U. A collection $\{(U_\alpha, \varphi_\alpha, \rho_\alpha)\}$ is called a **total trivialization atlas** for the vector bundle $E \to M$ provided that for each α, $(U_\alpha, \varphi_\alpha, \rho_\alpha)$ is a total trivialization triple and $\{(U_\alpha, \varphi_\alpha)\}$ is a smooth atlas for M. A collection $\{(U_\alpha, \varphi_\alpha, \rho_\alpha, \psi_\alpha)\}_{1 \leq \alpha \leq N}$ of 4-tuples is called an **augmented total trivialization atlas** for $E \to M$ provided that $\{(U_\alpha, \varphi_\alpha, \rho_\alpha)\}_{1 \leq \alpha \leq N}$ is a total trivialization atlas for $E \to M$ and $\{\psi_\alpha\}$ is a partition of unity subordinate to the open cover $\{U_\alpha\}$.

Definition 2.5. *Let M^n be a compact smooth manifold.*

- *We say that a total trivialization triple (U, φ, ρ) is **geometrically Lipschitz (GL)** provided that $\varphi(U)$ is a nonempty bounded open set with Lipschitz boundary. A total trivialization atlas is called **geometrically Lipschitz** if each of its total trivialization triples is GL.*

- *We say that a total trivialization triple (U,φ,ρ) is **nice** provided that $\varphi(U)$ is equal to a ball in $\mathbb{R}^n$. A total trivialization atlas is called nice if each of its total trivialization triples is nice.*
- *We say that a total trivialization triple (U,φ,ρ) is **super nice** provided that $\varphi(U)$ is equal to $\mathbb{R}^n$. A total trivialization atlas is called super nice if each of its total trivialization triples is super nice.*
- *A total trivialization atlas is called **generalized geometrically Lipschitz (GGL)** if each of its total trivialization triples is GL or super nice.*
- *We say that two total trivialization atlases $\{(U_\alpha,\varphi_\alpha,\rho_\alpha)\}_{\alpha\in I}$ and $\{(\tilde{U}_\beta,\tilde{\varphi}_\beta,\tilde{\rho}_\beta)\}_{\beta\in J}$ are **geometrically Lipschitz compatible (GLC)** if the corresponding atlases $\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in I}$ and $\{(\tilde{U}_\beta,\tilde{\varphi}_\beta)\}_{\beta\in J}$ are GLC.*

A *section* of a vector bundle $\pi: E\to M$ is a map $u: M\to E$ such that $\pi\circ u = Id_M$. We denote the space of all sections of E by $\Gamma(M,E)$. The space of all smooth sections of E is denoted by $C^\infty(M,E)$. In this manuscript, unless stated otherwise, a section of E refers to an element of $\Gamma(M,E)$ (no implicit smoothness assumption is made). Note that a section of the trivial vector bundle $E = M\times\mathbb{R}$ can be identified with a scalar function on M. In fact, $C^\infty(M, M\times\mathbb{R})$ can be identified with $C^\infty(M)$ where $C^\infty(M)$ is the collection of all smooth functions from M to $\mathbb{R}$. One can define sets of measure zero on a compact manifold using charts and it can be shown that such a definition is independent of the charts. In this manuscript, when we explicitly talk about the support of $u\in\Gamma(M,E)$ we mean the complement of the union of all open sets V in M such that $u=0$ almost everywhere on V. We are primarily interested in the bundle of $\binom{k}{l}$-tensors on M whose total space is

$$T^k_l(M) = \bigsqcup_{p\in M} T^k_l(T_pM)\,.$$

A section of this bundle is called a $\binom{k}{l}$-tensor field. We set $T^kM := T^k_0(M)$. TM denotes the tangent bundle of M and T^*M is the cotangent bundle of M. We set $\tau^k_l(M) = C^\infty(M, T^k_l(M))$ and $\chi(M) = C^\infty(M,TM)$.

For certain vector bundles there are standard methods to associate with any given smooth coordinate chart $(U,\varphi=(x^i))$ a total trivialization triple (U,φ,ρ). We call such a total trivialization triple the **standard total trivialization** associated with (U,φ). For example, consider $E = T^k_l(M)$. The collection of the following tensor fields on U form a local frame for E_U associated with $(U,\varphi=(x^i))$ in the sense that at each point $p\in U$, they form a basis for $T^k_l(T_pM)$:

$$\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_l}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_k}\,.$$

So, given any atlas $\{(U_\alpha,\varphi_\alpha)\}$ of a manifold M^n, there is a corresponding total trivialization atlas for the tensor bundle $T^k_l(M)$, namely $\{(U_\alpha,\varphi_\alpha,\rho_\alpha)\}$ where for each α, ρ_α has n^{k+l} components which we denote by $(\rho_\alpha)^{j_1\cdots j_l}_{i_1\cdots i_k}$. For all $F\in\Gamma(M,T^k_l(M))$, we have

$$(\rho_\alpha)^{j_1\cdots j_l}_{i_1\cdots i_k}(F) = (F_\alpha)^{j_1\cdots j_l}_{i_1\cdots i_k}\,.$$

Here $(F_\alpha)^{j_1\cdots j_l}_{i_1\cdots i_k}$ denotes the components of F with respect to the standard frame for $T^k_lU_\alpha$ described above. When there is no possibility of confusion, we may write $F^{j_1\cdots j_l}_{i_1\cdots i_k}$ instead of $(F_\alpha)^{j_1\cdots j_l}_{i_1\cdots i_k}$.

A symmetric positive definite section of T^2M is called a Riemannian metric on M. If M is equipped with a Riemannian metric g, the combination (M, g) will be referred to as a Riemannian manifold. For each $p \in M$, the norm induced by g on the tangent space T_pM will be denoted by $\|.\|_{g(p)}$ or just $\|.\|_g$. The corresponding operator norm for linear maps from T_pM to T_pM will be denoted by $\|.\|_{op(g(p))}$ or just $\|.\|_{op}$. We say that g is smooth (or the Riemannian manifold is smooth) if $g \in C^\infty(M, T^2M)$.

We denote the exterior derivative by d and grad : $C^\infty(M) \to \Gamma(M, TM)$ denotes the gradient operator which is defined by $g(\text{grad}\, f, X) = d\, f(X)$ for all $f \in C^\infty(M)$ and $X \in C^\infty(M, TM)$.

Given a metric g on M, one can define the musical isomorphisms as follows:

$$\begin{aligned} \text{flat}_g : T_pM &\to T_p^*M \\ X &\mapsto X^\flat := g(X, \cdot\,)\,, \\ \text{sharp}_g : T_p^*M &\to T_pM \\ \psi &\mapsto \psi^\sharp := \text{flat}_g^{-1}(\psi)\,. \end{aligned}$$

Using sharp_g we can define the $\binom{0}{2}$-tensor field g^{-1} (which is called the *inverse metric tensor*) as follows

$$g^{-1}(\psi_1, \psi_2) := g(\text{sharp}_g(\psi_1), \text{sharp}_g(\psi_2))\,.$$

Let $\{E_i\}$ be a local frame for the tangent bundle on an open subset $U \subset M$ and $\{\eta^i\}$ be the corresponding dual coframe. So, we can write $X = X^iE_i$ and $\psi = \psi_i\eta^i$. It is standard practice to denote the i^{th} component of flat_gX by X_i and the i^{th} component of $\text{sharp}_g(\psi)$ by ψ^i:

$$\text{flat}_gX = X_i\eta^i\,, \quad \text{sharp}_g\psi = \psi^iE_i\,.$$

It is easy to show that

$$X_i = g_{ij}X^j\,, \quad \psi^i = g^{ij}\psi_j\,,$$

where $g_{ij} = g(E_i, E_j)$ and $g^{ij} = g^{-1}(\eta^i, \eta^j)$. It is said that flat_gX is obtained from X by lowering an index and $\text{sharp}_g\psi$ is obtained from ψ by raising an index.

If (M, g) is a Riemannian manifold, then there exists a unique inner product on each fiber of $T^k_l(M)$ with the property that for all $x \in M$, if $\{e_i\}$ is an orthonormal basis of T_xM with dual basis $\{\eta^i\}$, then the corresponding basis of $T^k_l(T_xM)$ is orthonormal (see e.g. [17], page 29). We call this inner product the fiber metric on the bundle of $\binom{k}{l}$ tensors and denote it by $\langle ., .\rangle_F$. The corresponding norm is denoted by $|.|_F$. If A and B are two tensor fields, then with respect to any local frame

$$\langle A, B\rangle_F = g^{i_1r_1}\cdots g^{i_kr_k}g_{j_1s_1}\cdots g_{j_ls_l}A^{j_1\cdots j_l}_{i_1\cdots i_k}B^{s_1\cdots s_l}_{r_1\cdots r_k}\,.$$

Let (M^n, g) be a Riemannian manifold. Let $B : M \to \text{Hom}(TM, TM)$ be a continuous section of the vector bundle $\text{Hom}(TM, TM)$; in particular, for each $p \in M$, $B(p) : T_pM \to T_pM$ is a linear map. We define

$$\|B\|_\infty := \|f\|_{L^\infty(M)} = \sup_{p\in M} |f(p)|\,,$$

where the continuous function $f : M \to \mathbb{R}$ is defined by

$$f(p) = \|B(p)\|_{op(g(p))} \overset{\text{Theorem 3.3}}{=} \sup_{\|X\|_g = \|Y\|_g = 1} |g(BX, Y)| .$$

Note that, as a direct consequence of the above definition, for all $p \in M$ and $X, Y \in T_pM$ we have

$$|g(BX, Y)| \leq \|B\|_\infty \|X\|_g \|Y\|_g .$$

3. Background Material

Some background material on analysis, differential geometry, and function spaces and their properties is presented in this section. We simply state the basic results we need for the theorems we want to prove in the future sections. Almost all the theorems that are cited here, with proofs or appropriate references for the proofs, can be found in [7], [6], [4], and [11].

Theorem 3.1. *Let $(V, \langle ., . \rangle)$ be a finite dimensional (real) inner product space. If $B : V \times V \to \mathbb{R}$ is a bilinear form, then there exists a unique linear transformation $T : V \to V$ such that*

$$\forall\, x, y \in V \qquad B(x, y) = \langle T(x), y \rangle .$$

Moreover, if B is positive definite, then T is bijective. (Recall that a symmetric bilinear form B is called positive definite if $B(x, x) > 0$ for all nonzero x.)

Theorem 3.2. *([11], Page 154) Let $B : V \times V \to \mathbb{R}$ be a bilinear form on a normed space V and let Q be the associated quadratic form ($Q(x) = B(x, x)$). If B is symmetric and bounded, then $\|B\| = \|Q\|$, that is,*

$$\|B\| := \sup \{|B(x, y)| : \|x\| = \|y\| = 1\} = \sup \{|B(x, x)| : \|x\| = 1\} =: \|Q\| .$$

Theorem 3.3. *([11], Page 155) Let A be a bounded linear operator on a Hilbert space $(H, \langle ., . \rangle)$. Then the bilinear form defined by $B(x, y) = \langle Ax, y \rangle$ is bounded and $\|A\| = \|B\|$.*

Theorem 3.4. *Let X, Y, and Z be normed spaces. Suppose $A_n \to A$ in $L(X, Y)$ and $B_n \to B$ in $L(Y, Z)$. Then*

$$B_n \circ A_n \to B \circ A \qquad \text{in } L(X, Z) .$$

In particular, if $A_n \to A$ in $L(X, Y)$ and $B \in L(Y, Z)$, then $B \circ A_n \to B \circ A$.

Theorem 3.5. *Let $A : V \to W$ be a linear transformation between the normed spaces V and W. Then*

$$\|A\|_{op} = \sup_{\|x\|_V = 1,\, \|y\|_{W^*} = 1} |\langle y, Ax \rangle_{W^* \times W}| .$$

Proof. It is a direct consequence of Hahn-Banach theorem that for any $z \in W$, $\|w\|_W = \sup\{y(w) : y \in W^*, \ \|y\|_{W^*} = 1\}$ (see e.g. [8]). So,

$$\|A\|_{op} = \sup_{\|x\|_V = 1} \|Ax\|_W = \sup_{\|x\|_V = 1,\, \|y\|_{W^*} = 1} |\langle y, Ax \rangle_{W^* \times W}| .$$

□

Lemma 3.6. *([7], Page 20) Let M be a compact smooth manifold. Suppose $\{U_\alpha\}_{1 \leq \alpha \leq N}$ is an open cover of M. Suppose C is a closed set in M (so C is compact) which is contained in U_β for some $1 \leq \beta \leq N$. Then there exists a partition of unity $\{\psi_\alpha\}_{1 \leq \alpha \leq N}$ subordinate to $\{U_\alpha\}_{1 \leq \alpha \leq N}$ such that $\psi_\beta = 1$ on C.*

Theorem 3.7. *([7], Page 50) [Multiplication by smooth functions] Let $s \in \mathbb{R}$, $1 < p < \infty$, and $\varphi \in BC^\infty(\mathbb{R}^n)$. Then the linear map*

$$m_\varphi : W^{s,p}(\mathbb{R}^n) \to W^{s,p}(\mathbb{R}^n), \qquad u \mapsto \varphi u$$

is well-defined and bounded.

Theorem 3.8. *([7], Pages 54-55) Let Ω be a nonempty bounded open set in $\mathbb{R}^n$ with Lipschitz continuous boundary. Suppose $s \in \mathbb{R}$ and $p \in (1, \infty)$.*

(1) If $\varphi \in BC^\infty(\Omega)$, then the linear map $W^{s,p}(\Omega) \to W^{s,p}(\Omega)$ defined by $u \mapsto \varphi u$ is well-defined and bounded.

(2) Let $K \in \mathcal{K}(\Omega)$. If $\varphi \in C^\infty(\Omega)$, then the linear map $W^{s,p}_K(\Omega) \to W^{s,p}_K(\Omega)$ defined by $u \mapsto \varphi u$ is well-defined and bounded.

Theorem 3.9. *([7], Page 67) Let $s \in \mathbb{R}$, $1 < p < \infty$, and $\alpha \in \mathbb{N}_0^n$. Suppose Ω is a nonempty open set in $\mathbb{R}^n$. Then*

(1) the linear operator $\partial^\alpha : W^{s,p}(\mathbb{R}^n) \to W^{s-|\alpha|,p}(\mathbb{R}^n)$ is well-defined and bounded;

(2) for $s < 0$, the linear operator $\partial^\alpha : W^{s,p}(\Omega) \to W^{s-|\alpha|,p}(\Omega)$ is well-defined and bounded;

(3) for $s \geq 0$ and $|\alpha| \leq s$, the linear operator $\partial^\alpha : W^{s,p}(\Omega) \to W^{s-|\alpha|,p}(\Omega)$ is well-defined and bounded;

(4) if Ω is bounded with Lipschitz continuous boundary, and if $s \geq 0$, $s - \frac{1}{p} \neq$ integer (i.e. the fractional part of s is not equal to $\frac{1}{p}$), then the linear operator $\partial^\alpha : W^{s,p}(\Omega) \to W^{s-|\alpha|,p}(\Omega)$ for $|\alpha| > s$ is well-defined and bounded.

Theorem 3.10. *([6], Page 24) Let $s \in \mathbb{R}$, $1 < p < \infty$. Let Ω be a nonempty open set in $\mathbb{R}^n$. Either assume $\Omega = \mathbb{R}^n$ or Ω is Lipschitz or else assume s is not a noninteger less than -1. If A is a subset of $C_c^\infty(\Omega)$ with the following property:*

$$\forall\, x \in \Omega \quad \exists \varphi \in A \quad \text{such that} \quad \varphi \geq 0 \quad \text{and} \quad \varphi(x) \neq 0\,,$$

*then we say A is **admissible**. If A is an admissible family of functions then*

$$W^{s,p}_{loc}(\Omega) = \{u \in D'(\Omega) : \forall \varphi \in A \quad \varphi u \in W^{s,p}(\Omega)\}\,.$$

Theorem 3.11. *([6], Page 36) Let $s \in \mathbb{R}$, $1 < p < \infty$, and $\alpha \in \mathbb{N}_0^n$. Suppose Ω is a nonempty bounded open set in $\mathbb{R}^n$ with Lipschitz continuous boundary. Then*

(1) the linear operator $\partial^\alpha : W^{s,p}_{loc}(\mathbb{R}^n) \to W^{s-|\alpha|,p}_{loc}(\mathbb{R}^n)$ is well-defined and continuous;

(2) for $s < 0$, the linear operator $\partial^\alpha : W^{s,p}_{loc}(\Omega) \to W^{s-|\alpha|,p}_{loc}(\Omega)$ is well-defined and continuous;

(3) for $s \geq 0$ and $|\alpha| \leq s$, the linear operator $\partial^\alpha : W^{s,p}_{loc}(\Omega) \to W^{s-|\alpha|,p}_{loc}(\Omega)$ is well-defined and continuous;

(4) if $s \geq 0$, $s - \frac{1}{p} \neq$ integer (i.e. the fractional part of s is not equal to $\frac{1}{p}$), then the linear operator $\partial^\alpha : W^{s,p}_{loc}(\Omega) \to W^{s-|\alpha|,p}_{loc}(\Omega)$ for $|\alpha| > s$ is well-defined and continuous.

Theorem 3.12. *([4], Page 295, Page 298) Suppose $\Omega = \mathbb{R}^n$ or Ω is a bounded domain with Lipschitz continuous boundary. Assume s_i, s and $1 \leq p_i \leq p < \infty$ (i=1, 2) are real numbers satisfying*

- $s_i \geq s \geq 0$ *for* $i = 1, 2$,
- $s_i - s \geq n(\frac{1}{p_i} - \frac{1}{p})$,
- $s_1 + s_2 - s > n(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p})$.

Then, if $u \in W^{s_1,p_1}(\Omega)$ *and* $v \in W^{s_2,p_2}(\Omega)$, $uv \in W^{s,p}(\Omega)$*, and moreover, the pointwise multiplication of functions is a continuous bilinear map*

$$W^{s_1,p_1}(\Omega) \times W^{s_2,p_2}(\Omega) \to W^{s,p}(\Omega)\,.$$

Remark 3.13. *A number of other results concerning the sufficient conditions on the exponents* s_i, p_i, s, p *that guarantee the multiplication* $W^{s_1,p_1}(\Omega) \times W^{s_2,p_2}(\Omega) \hookrightarrow W^{s,p}(\Omega)$ *is well-defined and continuous are discussed in detail in* [4].

Theorem 3.14. *(*[6]*, Page 38) Let* $\Omega = \mathbb{R}^n$ *or* Ω *be a bounded open set in* $\mathbb{R}^n$ *with Lipschitz continuous boundary. Suppose* $s_1, s_2, s \in \mathbb{R}$ *and* $1 < p_1, p_2, p < \infty$ *are such that*

$$W^{s_1,p_1}(\Omega) \times W^{s_2,p_2}(\Omega) \hookrightarrow W^{s,p}(\Omega)\,.$$

(Here the symbol $\hookrightarrow$ *should be interpreted as described in Section 2). Then*

(1) $W^{s_1,p_1}_{loc}(\Omega) \times W^{s_2,p_2}_{loc}(\Omega) \hookrightarrow W^{s,p}_{loc}(\Omega)$.
(2) *For all* $K \in \mathcal{K}(\Omega)$, $W^{s_1,p_1}_{loc}(\Omega) \times W^{s_2,p_2}_{K}(\Omega) \hookrightarrow W^{s,p}(\Omega)$*. In particular, if* $f \in W^{s_1,p_1}_{loc}(\Omega)$*, then the mapping* $u \mapsto fu$ *is a well-defined continuous linear map from* $W^{s_2,p_2}_{K}(\Omega)$ *to* $W^{s,p}(\Omega)$.

Theorem 3.15. *(*[6]*, Pages 39-40) Let* Ω *be the same as the previous theorem. If* $sp > n$*, then* $W^{s,p}_{loc}(\Omega)$ *is closed under multiplication. Moreover, if*

$$(f_1)_m \to f_1 \quad \text{in } W^{s,p}_{loc}(\Omega), \cdots, (f_l)_m \to f_l \quad \text{in } W^{s,p}_{loc}(\Omega)\,,$$

then

$$(f_1)_m \cdots (f_l)_m \to f_1 \cdots f_l \quad \text{in } W^{s,p}_{loc}(\Omega)\,.$$

Theorem 3.16. *(*[6]*, Page 40) Let* $\Omega = \mathbb{R}^n$ *or let* Ω *be a nonempty bounded open set in* $\mathbb{R}^n$ *with Lipschitz continuous boundary. Let* $s \in \mathbb{R}$ *and* $p \in (1, \infty)$ *be such that* $sp > n$*. Let* $B : \Omega \to GL(k, \mathbb{R})$*. Suppose for all* $x \in \Omega$ *and* $1 \leq i, j \leq k$, $B_{ij}(x) \in W^{s,p}_{loc}(\Omega)$*. Then*

(1) *det* $B \in W^{s,p}_{loc}(\Omega)$.
(2) *Moreover if for each* $m \in \mathbb{N}$ $B_m : \Omega \to GL(k, \mathbb{R})$ *and for all* $1 \leq i, j \leq k$ $(B_m)_{ij} \to B_{ij}$ *in* $W^{s,p}_{loc}(\Omega)$*, then det* $B_m \to$ *det* B *in* $W^{s,p}_{loc}(\Omega)$.

Theorem 3.17. *(*[6]*, Page 40) Let* $\Omega = \mathbb{R}^n$ *or let* Ω *be a nonempty bounded open set in* $\mathbb{R}^n$ *with Lipschitz continuous boundary. Let* $s \geq 1$ *and* $p \in (1, \infty)$ *be such that* $sp > n$.

(1) *Suppose that* $u \in W^{s,p}_{loc}(\Omega)$ *and that* $u(x) \in I$ *for all* $x \in \Omega$ *where* I *is some interval in* $\mathbb{R}$*. If* $F : I \to \mathbb{R}$ *is a smooth function, then* $F(u) \in W^{s,p}_{loc}(\Omega)$.
(2) *Suppose that* $u_m \to u$ *in* $W^{s,p}_{loc}(\Omega)$ *and that for all* $m \geq 1$ *and* $x \in \Omega$, $u_m(x), u(x) \in I$ *where* I *is some open interval in* $\mathbb{R}$*. If* $F : I \to \mathbb{R}$ *is a smooth function, then* $F(u_m) \to F(u)$ *in* $W^{s,p}_{loc}(\Omega)$.
(3) *If* $F : \mathbb{R} \to \mathbb{R}$ *is a smooth function, then the map taking* u *to* $F(u)$ *is continuous from* $W^{s,p}_{loc}(\Omega)$ *to* $W^{s,p}_{loc}(\Omega)$.

Theorem 3.18. *(*[7]*, Page 22) Let* E *be a vector bundle of rank* r *over an* n*-dimensional compact smooth manifold* M*. Then* E *admits a finite total trivialization atlas that is GL compatible with itself. In fact, there exists a total trivialization atlas* $\{(U_\alpha, \varphi_\alpha, \rho_\alpha)\}_{1 \leq \alpha \leq N}$ *such that*

- *for all* $1 \leq \alpha \leq N$, $\varphi_\alpha(U_\alpha)$ *is bounded with Lipschitz continuous boundary, and,*
- *for all* $1 \leq \alpha, \beta \leq N$, $U_\alpha \cap U_\beta$ *is either empty or else* $\varphi_\alpha(U_\alpha \cap U_\beta)$ *and* $\varphi_\beta(U_\alpha \cap U_\beta)$ *are bounded with Lipschitz continuous boundary.*

Definition 3.19. *Let* M^n *be a compact smooth manifold. Let* $\pi : E \to M$ *be a vector bundle of rank* r. *Let* $\Lambda = \{(U_\alpha, \varphi_\alpha, \rho_\alpha, \psi_\alpha)\}_{1\leq\alpha\leq N}$ *be an augmented total trivialization atlas for* $E \to M$. *For* $e \in \mathbb{R}$ *and* $q \in (1, \infty)$, $W^{e,q}(M, E; \Lambda)$ *is defined as the completion of* $C^\infty(M, E)$ *with respect to the norm*

$$\|u\|_{W^{e,q}(M,E;\Lambda)} = \sum_{\alpha=1}^{N} \sum_{l=1}^{r} \|(\rho_\alpha)^l \circ (\psi_\alpha u) \circ \varphi_\alpha^{-1}\|_{W^{e,q}(\varphi_\alpha(U_\alpha))} .$$

It can be proved that if e is not a noninteger less than -1 (that is, $e \notin (-\infty, -1) \setminus \mathbb{Z}$), the above definition is independent of the choice of the total trivialization atlas. Also if e is a noninteger less than -1 (that is, $e \in (-\infty, -1) \setminus \mathbb{Z}$), the definition does not depend on Λ as long as it is assumed that Λ is GL compatible with itself (see e.g. [7] for detailed discussion). So, we set $W^{e,q}(M, E) := W^{e,q}(M, E; \Lambda)$ where if $e \notin (-\infty, -1)\setminus \mathbb{Z}$, Λ is any augmented total trivialization atlas, and if $e \in (-\infty, -1) \setminus \mathbb{Z}$, Λ is any augmented total trivialization atlas that is GL compatible with itself. Sometimes, instead of $W^{e,q}(M, E)$, we may just write $W^{e,q}(E)$.

Theorem 3.20. *([7], Page 83) Let* M^n *be a compact smooth manifold and* $E \to M$ *be a vector bundle of rank* r. *Suppose* $\Lambda = \{(U_\alpha, \varphi_\alpha, \rho_\alpha, \psi_\alpha)\}_{\alpha=1}^N$ *is an augmented total trivialization atlas for* $E \to M$. *Let* u *be a section of* E, $e \in \mathbb{R}$, *and* $q \in (1, \infty)$. *If for all* $1 \leq \alpha \leq N$ *and* $1 \leq j \leq r$, $(\rho_\alpha)^j \circ u \circ \varphi_\alpha^{-1} \in W^{e,q}_{loc}(\varphi_\alpha(U_\alpha))$, *then* $u \in W^{e,q}(M, E; \Lambda)$.

Theorem 3.21. *([7], Page 84) Let* M^n *be a compact smooth manifold and* $E \to M$ *be a vector bundle of rank* r. *Let* $e \in \mathbb{R}$ *and* $q \in (1, \infty)$. *Suppose* $\Lambda = \{(U_\alpha, \varphi_\alpha, \rho_\alpha, \psi_\alpha)\}_{\alpha=1}^N$ *is an augmented total trivialization atlas for* $E \to M$. *If* e *is a noninteger less than* -1 *further assume that* Λ *is GL compatible with itself. If a section* u *of the vector bundle* E *belongs to* $W^{e,q}(M, E; \Lambda)$, *then for all* $1 \leq \alpha \leq N$ *and* $1 \leq i \leq r$, $(\rho_\alpha)^i \circ u \circ \varphi_\alpha^{-1}$ *(i.e. each component of the local representation of* u *with respect to* $(U_\alpha, \varphi_\alpha, \rho_\alpha)$*) belongs to* $W^{e,q}_{loc}(\varphi_\alpha(U_\alpha))$. *Moreover, if* $\xi \in C_c^\infty(\varphi_\alpha(U_\alpha))$, *then*

$$\|\xi\big((\rho_\alpha)^j \circ u \circ \varphi_\alpha^{-1}\big)\|_{W^{e,q}(\varphi_\alpha(U_\alpha))} \preceq \|u\|_{W^{e,q}(M,E;\Lambda)} ,$$

where the implicit constant may depend on ξ *but does not depend on* u.

Theorem 3.22. *([7], Page 81) Let* M^n *be a compact smooth manifold. Let* $\pi : E \to M$ *be a smooth vector bundle of rank* r *over* M *equipped with fiber metric* $\langle ., . \rangle_E$ *(so it is meaningful to talk about* $L^\infty(M, E)$*). Suppose* $s \in \mathbb{R}$ *and* $p \in (1, \infty)$ *are such that* $sp > n$. *Then* $W^{s,p}(M, E) \hookrightarrow L^\infty(M, E)$. *Moreover, every element* u *in* $W^{s,p}(M, E)$ *has a continuous representative.*

The following corollary is an immediate consequence of Theorem 3.21.

Corollary 3.23. *Let* (M^n, g) *be a compact Riemannian manifold with* $g \in W^{s,p}(T^2M)$, $sp > n$. *Let* $\{(U_\alpha, \varphi_\alpha, \rho_\alpha)\}_{1\leq\alpha\leq N}$ *be a standard total trivialization atlas for* $T^2M \to M$. *Fix some* α *and denote the components of the metric with respect to* $(U_\alpha, \varphi_\alpha, \rho_\alpha)$ *by* $g_{ij} : U_\alpha \to \mathbb{R}$ $(g_{ij} = (\rho_\alpha)_{ij} \circ g)$. *Then*

$$g_{ij} \circ \varphi_\alpha^{-1} \in W^{s,p}_{loc}(\varphi_\alpha(U_\alpha)) .$$

Theorem 3.24. *([7], Page 85) Let (M^n, g) be a compact Riemannian manifold with $g \in W^{s,p}(T^2M)$, $sp > n$, $s \geq 1$. Let $\{(U_\alpha, \varphi_\alpha, \rho_\alpha)\}_{1\leq\alpha\leq N}$ be a GGL standard total trivialization atlas for $T^2M \to M$. Fix some α and denote the components of the metric with respect to $(U_\alpha, \varphi_\alpha, \rho_\alpha)$ by $g_{ij} : U_\alpha \to \mathbb{R}$ ($g_{ij} = (\rho_\alpha)_{ij} \circ g$). Then*

(1) $\det g_\alpha \in W^{s,p}_{loc}(\varphi_\alpha(U_\alpha))$ where $g_\alpha(x)$ is the matrix whose (i, j)-entry is $g_{ij} \circ \varphi_\alpha^{-1}$.
(2) $\sqrt{\det g} \circ \varphi_\alpha^{-1} = \sqrt{\det g_\alpha} \in W^{s,p}_{loc}(\varphi_\alpha(U_\alpha))$.
(3) $\frac{1}{\sqrt{\det g \circ \varphi_\alpha^{-1}}} \in W^{s,p}_{loc}(\varphi_\alpha(U_\alpha))$.

Theorem 3.25. *([7], Pages 85-86) Let (M^n, g) be a compact Riemannian manifold with $g \in W^{s,p}(T^2M)$, $sp > n$, $s \geq 1$. $\{(U_\alpha, \varphi_\alpha)\}_{1\leq\alpha\leq N}$ be a GGL smooth atlas for M. Denote the standard components of the inverse metric with respect to this chart by $g^{ij} : U_\alpha \to \mathbb{R}$. Then*

$$g^{ij} \circ \varphi_\alpha^{-1} \in W^{s,p}_{loc}(\varphi_\alpha(U_\alpha)) \, ,$$

moreover,

$$\Gamma^k_{ij} \circ \varphi_\alpha^{-1} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \circ \varphi_\alpha^{-1} \in W^{s-1,p}_{loc}(\varphi_\alpha(U_\alpha)) \, .$$

(Γ^k_{ij}'s denote the Christoffel symbols.)

Theorem 3.26. *([7], Page 91) Let M^n be a compact smooth manifold and let $\pi : E \to M$ be a vector bundle of rank r equipped with a fiber metric $\langle ., . \rangle_E$. Let $e \in \mathbb{R}$ and $q \in (1, \infty)$. Suppose $\Lambda = \{(U_\alpha, \varphi_\alpha, \rho_\alpha, \psi_\alpha)\}_{\alpha=1}^N$ is an augmented total trivialization atlas for $E \to M$. If e is a noninteger whose magnitude is greater than 1 further assume that the total trivialization atlas in Λ is GL compatible with itself. Fix a positive smooth density μ on M.*
Consider the L^2 inner product on $C^\infty(M, E)$ defined by

$$\langle u, v \rangle_2 = \int_M \langle u, v \rangle_E \mu \, .$$

Then

(i) $\langle ., . \rangle_2$ extends uniquely to a continuous bilinear pairing $\langle ., . \rangle_2 : W^{-e,q'}(M, E; \Lambda) \times W^{e,q}(M, E; \Lambda) \to \mathbb{R}$. (We are using the same notation (i.e. $\langle ., . \rangle_2$) for the extended bilinear map!)
(ii) The map $S : W^{-e,q'}(M, E, \Lambda) \to [W^{e,q}(M, E; \Lambda)]^*$ defined by $S(u) = l_u$ where

$$l_u : W^{e,q}(M, E; \Lambda) \to \mathbb{R}, \quad l_u(v) = \langle u, v \rangle_2$$

is a well-defined topological isomorphism.

In particular, $[W^{e,q}(M, E; \Lambda)]^*$ can be identified with $W^{-e,q'}(M, E; \Lambda)$.

4. Preliminary Results

Suppose (M^n, g) is a compact Riemannian manifold with $g \in W^{s,p}(T^2M)$, $sp > n$, and $s \geq 1$. Suppose $\{g_m\}$ is a sequence of smooth metrics that converges to g in $W^{s,p}(T^2M)$. In this section we go over some of the immediate consequences of this assumption which will be useful in the study of the main results presented in this work. As it was pointed out in the introduction, the ultimate goal of this manuscript is to study the relationship between various geometric operators (like Laplacian) that are defined in terms of g_m's and those that are defined in terms of g. We will present two rather distinct methods to accomplish this goal:

(1) The first approach works for a limited range of Sobolev spaces and follows (and extends) the argument presented in [12] for the Laplace operator with the domain $H^1(M) = W^{1,2}(M)$. This method is based on the notion of "metric distortion tensor" and duality arguments.

(2) The second approach works for a wider range of Sobolev spaces and will be based on the previously mentioned characterization of Sobolev spaces in terms of local coordinates and theorems on multiplication properties of Sobolev spaces and behavior of Sobolev functions under composition.

Let's begin with the notion of metric distortion tensor. By Theorem 3.1 for each m and at each $p \in M$ there exists a linear operator $A_m|_p : T_pM \to T_pM$ (when the basepoint is clear from the context instead of $A_m|_p$ we just write A_m) such that

$$\forall\, X, Y \in T_pM \qquad g_m(X,Y) = g(A_m X, Y)\,.$$

A_m is called the **metric distortion tensor** associated with g_m (see [12] and [13]). A_m can be viewed as a continuous section of the bundle Hom(TM, TM); we have

$$\|A_m - Id\|_\infty = \Big\| \sup_{\|X\|_g = \|Y\|_g = 1} |g((A_m - Id)X, Y))| \Big\|_{L^\infty(M)}\,,$$

where $Id_p : T_pM \to T_pM$ is the identity map. In particular, note that for all $p \in M$ and $X, Y \in T_pM$

$$|g((A_m - Id)X, Y))| \le \|A_m - Id\|_\infty \|X\|_g \, \|Y\|_g\,.$$

The following two theorems play a key role in the first approach mentioned above.

Theorem 4.1. *Let M^n be a compact smooth manifold equipped with a Riemannian metric g. Denote the norm induced by the fiber metric on the bundle of $\binom{2}{0}$ tensors by $|.|_F$. If S is a symmetric covariant tensor field of order 2, then*

$$\forall\, p \in M \qquad \sup\big\{|S_p(X,Y)| : X, Y \in T_pM,\ \|X\|_g = \|Y\|_g = 1\big\} \le |S|_F(p)\,.$$

Note that the left hand side of the above inequality is the norm of S_p as a bilinear form on the inner product space (T_pM, g_p).

Proof. In this proof we will not use the summation convention. Let $p \in M$. Let $\{E_i\}$ be an orthonormal basis for T_pM. At p the components of the metric with respect to $\{E_i\}$ are given by $g_{ij} = \delta_{ij}$. We have

$$|S|_F^2(p) = \sum_{i,j,r,s} g^{ir} g^{js} S_{ij} S_{rs}|_p = \sum_{i,j,r,s} \delta^{ir}\delta^{js} S_{ij} S_{rs}|_p = \sum_{i=1}^{n}\sum_{j=1}^{n} S_{ij}^2(p)\,.$$

Now, let $A : T_pM \to T_pM$ be the unique linear transformation such that (see Theorem 3.1)

$$\forall\, X, Y \in T_pM \qquad S_p(X,Y) = g_p(AX, Y)\,.$$

If $X \in T_pM$ is such that $\|X\|_g = 1$ (note that since $g = \delta$ at p, we have $\|X\|_g = \sum_{i,j} g_{ij}X^iX^j = \sum_{i=1}^n |X^i|^2$), then

$$\begin{aligned}
|S_p(X,X)|^2 &= |g_p(AX,X)|^2 \le \|AX\|_g^2\|X\|_g^2\\
&= \|AX\|_g^2 = \Big\| \sum_{i=1}^n X^i(AE_i)\Big\|_g^2\\
&\le \Big(\sum_{i=1}^n |X^i|\|AE_i\|_g\Big)^2 \le \Big(\sum_{i=1}^n |X^i|^2\Big)\Big(\sum_{i=1}^n \|AE_i\|_g^2\Big)\\
&= \sum_{i=1}^n \|AE_i\|_g^2 = \sum_{i=1}^n\sum_{j=1}^n g_p(AE_i,E_j)^2 = \sum_{i=1}^n\sum_{j=1}^n S_{ij}^2(p) = |S|_F^2(p)\,.
\end{aligned}$$

Note that we used the fact that since $\{E_i\}$ is orthonormal

$$AE_i = \sum_{j=1}^n g_p(AE_i,E_j)E_j \Longrightarrow \|AE_i\|_g^2 = \sum_{j=1}^n g_p(AE_i,E_j)^2\,.$$

Therefore,

$$\begin{aligned}
&\sup\big\{|S_p(X,Y)| : X,Y \in T_pM,\ \|X\|_g = \|Y\|_g = 1\big\}\\
&\qquad \overset{\text{Theorem 3.2}}{=} \sup\big\{|S_p(X,X)| : X \in T_pM,\ \|X\|_g = 1\big\}\\
&\qquad \le |S|_F(p)\,.
\end{aligned}$$

□

Theorem 4.2. *Let M^n be a compact smooth manifold. Let $\{g_m\}$ be a sequence of smooth metrics on M. Let $g \in \Gamma(M,T^2M)$ be a metric on M that belongs to $W^{s,p}(T^2M)$ with $sp > n$ and $s \ge 1$. Suppose $g_m \to g$ in $W^{s,p}(T^2M)$. Denote the metric distortion tensor associated with g_m by A_m. Then*

(1) $\|A_m - Id\|_\infty \preceq \|g_m - g\|_{s,p}$. As a result, since $\|g_m - g\|_{s,p} \to 0$, we have $\|A_m - Id\|_\infty \to 0$.

(2) If $\|A_m - Id\|_\infty \to 0$, then $\|A_m^{-1} - Id\|_\infty \to 0$,

(3) If $\|A_m - Id\|_\infty \to 0$ and $det\, A_m \to 1$ uniformly on M, then $\|\sqrt{det\, A_m}A_m^{-1} - Id\|_\infty \to 0$.

(4) $A_m^{-1} grad_g = grad_{g_m}$.

(5) $dV_{g_m} = \sqrt{det\, A_m} dV_g$ (dV_g denotes the Riemannian density with respect to the metric g).

Proof.

(1) Denote the norm induced by the corresponding fiber metric (associated with the metric g) on the bundle of $\binom{2}{0}$ tensors by $|.|_F$.

$$
\begin{aligned}
\|g_m - g\|_{s,p} \succeq \|g_m - g\|_{L^\infty(T^2M)} &= \big\| |g_m - g|_F \big\|_{L^\infty(M)} \quad (W^{s,p} \hookrightarrow L^\infty) \\
&\geq \Big\| \sup_{\|X\|_g = \|Y\|_g = 1} |g_m(X,Y) - g(X,Y)| \Big\|_{L^\infty(M)} \qquad \text{(see Theorem 4.1)} \\
&= \Big\| \sup_{\|X\|_g = \|Y\|_g = 1} |g(A_m X,Y) - g(X,Y)| \Big\|_{L^\infty(M)} \\
&= \Big\| \sup_{\|X\|_g = \|Y\|_g = 1} |g((A_m - Id)X,Y)| \Big\|_{L^\infty(M)} \\
&= \|A_m - Id\|_\infty \,.
\end{aligned}
$$

(2) By assumption, $\sup_{x\in M} \|A_m(x) - Id\|_{op(g(x))} \to 0$. Therefore, there exists $m_0 \in \mathbb{N}$ such that

$$
\forall\, m \geq m_0 \qquad \forall\, x \in M \qquad\qquad \|A_m(x) - Id\|_{op(g(x))} < 1\,.
$$

Note that, as a consequence, for all $m \geq m_0$ and $x \in M$, we have $A_m^{-1}(x) = \sum_{k=0}^{\infty} (Id - A_m(x))^k$, and so

$$
\|A_m(x)^{-1}\|_{op(g(x))} \leq \sum_{k=0}^{\infty} \|Id - A_m(x)\|^k_{op(g(x))} = \frac{1}{1 - \|Id - A_m(x)\|_{op(g(x))}}\,.
$$

Therefore, for all $m \geq m_0$,

$$
\begin{aligned}
\|A_m^{-1} - Id\|_\infty &= \sup_{x\in M} \|A_m(x)^{-1} - Id\|_{op(g(x))} = \sup_{x\in M} \|A_m(x)^{-1}\big(Id - A_m(x)\big)\|_{op(g(x))} \\
&\leq \sup_{x\in M} \Big(\|A_m(x)^{-1}\|_{op(g(x))} \|Id - A_m(x)\|_{op(g(x))} \Big) \\
&\leq \Big(\sup_{x\in M} \frac{1}{1 - \|Id - A_m(x)\|_{op(g(x))}} \Big) \Big(\sup_{x\in M} \|Id - A_m(x)\|_{op(g(x))} \Big) \to 0\,.
\end{aligned}
$$

(3) Note that for each m, the function $x \mapsto \det A_m(x)$ is a continuous function from M to $\mathbb{R}$. Therefore, since M is compact, for each m, $\sup_{x\in M} \det A_m(x)$ is finite. This together with the assumption that $\det A_m(x)$ converges uniformly to 1 imply that the functions $\{x \mapsto \det A_m(x)\}_{x\in M,\, m\in\mathbb{N}}$ are uniformly bounded. That is, there exists $R > 0$ such that

$$
\forall\, m \in \mathbb{N} \quad \forall\, x \in M \qquad\qquad \det A_m(x) \leq R\,.
$$

Also, note that since $\det A_m(x) \to 1$ uniformly and square root is uniformly continuous, we have $\sqrt{\det A_m(x)} \to 1$ uniformly. Hence we can write

$$
\begin{aligned}
&\sup_{x\in M} \| \sqrt{\det A_m} A_m^{-1} - Id\|_{op(g(x))} \\
&\quad \leq \sup_{x\in M} \big(\|\sqrt{\det A_m} A_m^{-1} - \sqrt{\det A_m} Id\|_{op(g(x))} + \|\sqrt{\det A_m} Id - Id\|_{op(g(x))} \big) \\
&\quad \leq \sup_{x\in M} \big(\sqrt{\det A_m} \|A_m^{-1} - Id\|_{op(g(x))} \big) + \sup_{x\in M} \big((\sqrt{\det A_m} - 1) \|Id\|_{op(g(x))} \big) \\
&\quad \leq \sqrt{R} \sup_{x\in M} \|A_m^{-1} - Id\|_{op(g(x))} + \sup_{x\in M} (\sqrt{\det A_m} - 1) \to 0\,.
\end{aligned}
$$

Items (4) and (5) are direct consequences of the definition of grad and the standard expression for dV in each coordinate neighborhood. □

The next theorem plays an important role in the second approach that was mentioned in the beginning of this section.

Theorem 4.3. *Let (M^n, g) be a Riemannian manifold. Let $\{g_m\}$ be a sequence of smooth metrics on M. Suppose $g_m \to g$ in $W^{s,p}(T^2M)$ with $sp > n$ and $s \geq 1$. Let $\{(U_\alpha, \varphi_\alpha, \rho_\alpha)\}_{1\leq\alpha\leq N}$ and $\{(U_\alpha, \varphi_\alpha, \tilde{\rho}_\alpha)\}_{1\leq\alpha\leq N}$ be GGL standard total trivialization atlases for T^2M and T_2M, respectively. Then*

(1) For all $1 \leq \alpha \leq N$, $1 \leq i, j \leq n$: $(g_m)_{ij} \circ \varphi_\alpha^{-1} \to g_{ij} \circ \varphi_\alpha^{-1}$ in $W^{s,p}_{loc}(\varphi_\alpha(U_\alpha))$.
(2) For all $1 \leq \alpha \leq N$, $1 \leq i, j \leq n$: $(g_m)^{ij} \circ \varphi_\alpha^{-1} \to g^{ij} \circ \varphi_\alpha^{-1}$ in $W^{s,p}_{loc}(\varphi_\alpha(U_\alpha))$.
(3) $(g_m)^{-1} \to g^{-1}$ in $W^{s,p}(T_2M)$.
(4) For all $1 \leq i, j, k \leq n$: $(\Gamma_{g_m})^k_{ij} \circ \varphi_\alpha^{-1} \to (\Gamma_g)^k_{ij} \circ \varphi_\alpha^{-1}$ in $W^{s-1,p}_{loc}(\varphi_\alpha(U_\alpha))$.

Proof. First let us define a suitable family of *admissible* test functions (see Theorem 3.10) on $\varphi_\alpha(U_\alpha)$. For each $x \in \varphi_\alpha(U_\alpha)$, choose $r_x > 0$ such that

$$\bar{B}_{r_x}(x) \subseteq \varphi_\alpha(U_\alpha)\,.$$

Let $V_x = \varphi_\alpha^{-1}(B_{r_x}(x))$. Clearly $V_x \subseteq \bar{V}_x \subseteq U_\alpha$. Therefore, by Lemma 3.6 there exists a partition of unity $\{\psi_{\beta,x}\}$ subordinate to $\{U_\beta\}_{1\leq\beta\leq N}$ such that $\psi_{\alpha,x} = 1$ on $\bar{V}_x$. We define $\tilde{\psi}_x = \psi_{\alpha,x} \circ \varphi_\alpha^{-1}$. $\{\tilde{\psi}_x\}_{x\in\varphi_\alpha(U_\alpha)}$ is an admissible family of test functions on $\varphi_\alpha(U_\alpha)$. So, in order to prove that a sequence $\{f_m\}$ converges to f in $W^{s,p}_{loc}(\varphi_\alpha(U_\alpha))$ it is enough to show that

$$\forall\, x \in \varphi_\alpha(U_\alpha) \qquad \tilde{\psi}_x f_m \to \tilde{\psi}_x f \quad \text{in } W^{s,p}(\varphi_\alpha(U_\alpha))\,.$$

(1) Let $x \in \varphi_\alpha(U_\alpha)$. We have

$$\begin{aligned}\|g_m - g\|_{s,p} &\simeq \sum_{\beta=1}^{N}\sum_{i,j=1}^{n} \|(\rho_\beta)_{ij} \circ (\psi_{\beta,x}(g_m - g)) \circ \varphi_\beta^{-1}\|_{W^{s,p}(\varphi_\beta(U_\beta))} \\ &\simeq \sum_{\beta=1}^{N}\sum_{i,j=1}^{n} \|\psi_{\beta,x}[(g_m)_{ij} - g_{ij}] \circ \varphi_\beta^{-1}\|_{W^{s,p}(\varphi_\beta(U_\beta))}\,.\end{aligned}$$

By assumption $\|g_m - g\|_{s,p} \to 0$ and so

$$\forall\, 1 \leq \beta \leq N \quad \forall\, 1 \leq i, j \leq n \qquad \|\psi_{\beta,x}[(g_m)_{ij} - g_{ij}] \circ \varphi_\beta^{-1}\|_{W^{s,p}(\varphi_\beta(U_\beta))} \to 0\,.$$

In particular,

$$\forall\, 1 \leq i, j \leq n \qquad \|\psi_{\alpha,x}[(g_m)_{ij} - g_{ij}] \circ \varphi_\alpha^{-1}\|_{W^{s,p}(\varphi_\alpha(U_\alpha))} \to 0\,.$$

Considering that $\psi_{\alpha,x} \circ \varphi_\alpha^{-1} = \tilde{\psi}_x$, we get

$$\forall\, 1 \leq i, j \leq n \qquad \|\tilde{\psi}_x[((g_m)_{ij} - g_{ij}) \circ \varphi_\alpha^{-1}]\|_{W^{s,p}(\varphi_\alpha(U_\alpha))} \to 0\,.$$

Since $x \in \varphi_\alpha(U_\alpha)$ is arbitrary and $\{\tilde{\psi}_y\}_{y\in\varphi_\alpha(U_\alpha)}$ form an admissible family of test functions, we can conclude that

$$(g_m)_{ij} \circ \varphi_\alpha^{-1} \to g_{ij} \circ \varphi_\alpha^{-1} \quad \text{in } W^{s,p}_{loc}(\varphi_\alpha(U_\alpha)).$$

(2) Let $C = (C_{ij})$ and $C_m = ((C_m)_{ij})$ where $C_{ij} = g_{ij} \circ \varphi_\alpha^{-1}$ and $(C_m)_{ij} = (g_m)_{ij} \circ \varphi_\alpha^{-1}$. Our goal is to show that

$$(C_m^{-1})_{ij} \to (C^{-1})_{ij} \quad \text{in } W^{s,p}_{loc}(\varphi_\alpha(U_\alpha))\,.$$

Recall that

$$(C^{-1})_{ij} = \frac{(-1)^{i+j}}{\det C} M_{ij}\,,$$
$$((C_m)^{-1})_{ij} = \frac{(-1)^{i+j}}{\det C_m} (M_m)_{ij}\,.$$

where M_{ij} and $(M_m)_{ij}$ are the determinants of the $(n-1)\times(n-1)$ matrices formed by removing the j^{th} row and i^{th} column of C and C_m, respectively. By item 1 we know that $(C_m)_{ij} \to C_{ij}$ in $W^{s,p}_{loc}(\varphi_\alpha(U_\alpha))$. So, it follows from Theorem 3.16 that

$$\det C_m \to \det C, \qquad (M_m)_{ij} \to M_{ij} \quad \text{in } W^{s,p}_{loc}(\varphi_\alpha(U_\alpha))\,.$$

As a direct consequence of Theorem 3.17,

$$\frac{1}{\det C_m} \to \frac{1}{\det C} \quad \text{in } W^{s,p}_{loc}(\varphi_\alpha(U_\alpha))\,.$$

Hence by Theorem 3.14 and Theorem 3.15 we can conclude that

$$\frac{(-1)^{i+j}}{\det C_m}(M_m)_{ij} \to \frac{(-1)^{i+j}}{\det C} M_{ij} \quad \text{in } W^{s,p}_{loc}(\varphi_\alpha(U_\alpha))\,.$$

(3) Let $\{\theta_\beta\}_{1\leq\beta\leq N}$ be a partition of unity subordinate to $\{U_\beta\}_{1\leq\beta\leq N}$. We have

$$\|(g_m)^{-1} - g^{-1}\|_{s,p} \simeq \sum_{\beta=1}^{N}\sum_{i,j=1}^{n} \|(\theta_\beta((g_m)^{ij} - g^{ij}))\circ\varphi_\beta^{-1}\|_{W^{s,p}(\varphi_\beta(U_\beta))}\,.$$

According to item 2, for all $1\leq\beta\leq N$,

$$(g_m)^{ij}\circ\varphi_\beta^{-1} \to g^{ij}\circ\varphi_\beta^{-1} \quad \text{in } W^{s,p}_{loc}(\varphi_\beta(U_\beta))\,.$$

Therefore, it follows from the definition of convergence in $W^{s,p}_{loc}(\varphi_\beta(U_\beta))$ that

$$\|(\theta_\beta((g_m)^{ij} - g^{ij}))\circ\varphi_\beta^{-1}\|_{W^{s,p}(\varphi_\beta(U_\beta))} \to 0\,.$$

Hence $\|(g_m)^{-1} - g^{-1}\|_{s,p} \to 0$.

(4) Recall that

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})\,,$$
$$(\Gamma_m)^k_{ij} = \frac{1}{2}(g_m)^{kl}(\partial_i (g_m)_{jl} + \partial_j (g_m)_{il} - \partial_l (g_m)_{ij})\,.$$

By item 1 and item 2 we have

$$(g_m)^{kl} \to g^{kl}, \quad (g_m)_{jl} \to g_{jl}, \quad (g_m)_{il} \to g_{il}, \quad (g_m)_{ij} \to g_{ij} \quad \text{in } W^{s,p}_{loc}(\varphi_\alpha(U_\alpha))\,.$$

By Theorem 3.11 partial differentiation with respect to any one of the variables is continuous from $W^{s,p}_{loc}(\varphi_\alpha(U_\alpha))$ to $W^{s-1,p}_{loc}(\varphi_\alpha(U_\alpha))$. Also, it follows from Theorem 3.14 that

$$W^{s,p}_{loc}(\varphi_\alpha(U_\alpha)) \times W^{s-1,p}_{loc}(\varphi_\alpha(U_\alpha)) \hookrightarrow W^{s-1,p}_{loc}(\varphi_\alpha(U_\alpha))\,.$$

The claim of this item is a direct consequence of the above observations.

□

5. Sharp Operator with Rough Metric

Theorem 5.1. *Let (M^n, g) be a compact Riemannian manifold. Assume $g \in W^{s,p}(T^2M)$ with $sp > n$ and $s \geq 1$. Suppose $e \in \mathbb{R}$ and $q \in (1,\infty)$ are such that for balls $\Omega \subseteq \mathbb{R}^n$ or for $\Omega = \mathbb{R}^n$*

$$W^{s,p}(\Omega) \times W^{e,q}(\Omega) \hookrightarrow W^{e,q}(\Omega)\,.$$

Then $sharp_g : (C^\infty(M, T^*M), \|.\|_{e,q}) \to W^{e,q}(TM)$ is continuous and so it has a unique extension to a continuous operator $sharp_g : W^{e,q}(T^*M) \to W^{e,q}(TM)$.

Proof. Let $\Lambda = \{(U_\alpha, \varphi_\alpha, \rho_\alpha)\}_{\alpha=1}^N$ be a standard total trivialization atlas for TM and $\tilde{\Lambda} = \{(U_\alpha, \varphi_\alpha, \tilde{\rho}_\alpha)\}_{\alpha=1}^N$ be a standard total trivialization atlas for T^*M. Without loss of generality we may assume that each of Λ and $\tilde{\Lambda}$ is nice (or super nice) and GL compatible with itself (see Theorem 3.18 and [7]). Let $\{\psi_\alpha\}_{\alpha=1}^N$ be a partition of unity subordinate to the open cover $\{U_\alpha\}_{\alpha=1}^N$. Let $\tilde{\psi}_\alpha = \frac{\psi_\alpha^2}{\sum_{\beta=1}^N \psi_\beta^2}$. Note that $\frac{1}{\sum_{\beta=1}^N \psi_\beta^2} \circ \varphi_\alpha^{-1} \in BC^\infty(\varphi_\alpha(U_\alpha))$. We have

$$\begin{aligned}
\|\text{sharp}_g\omega\|_{W^{e,q}(TM)} &\simeq \sum_{\alpha=1}^N \sum_{i=1}^n \|\tilde{\psi}_\alpha(\rho_\alpha)^i(\text{sharp}_g\omega) \circ \varphi_\alpha^{-1}\|_{W^{e,q}(\varphi_\alpha(U_\alpha))} \\
&\simeq \sum_{\alpha=1}^N \sum_{i=1}^n \|\tilde{\psi}_\alpha g^{ij}\omega_j \circ \varphi_\alpha^{-1}\|_{W^{e,q}(\varphi_\alpha(U_\alpha))} \\
&\preceq \sum_{\alpha=1}^N \sum_{i=1}^n \|\psi_\alpha^2 g^{ij}\omega_j \circ \varphi_\alpha^{-1}\|_{W^{e,q}(\varphi_\alpha(U_\alpha))} \\
&\preceq \sum_{\alpha=1}^N \sum_{i=1}^n \sum_{j=1}^n \|\psi_\alpha g^{ij} \circ \varphi_\alpha^{-1}\|_{W^{s,p}(\varphi_\alpha(U_\alpha))} \|\psi_\alpha\omega_j \circ \varphi_\alpha^{-1}\|_{W^{e,q}(\varphi_\alpha(U_\alpha))} \\
&\preceq \|g^{-1}\|_{W^{s,p}(T^0_2M)} \|\omega\|_{W^{e,q}(T^*M)}\,.
\end{aligned}$$

Note that the inequality in the third line follows from Theorem 3.7 and Theorem 3.8. ☐

Theorem 5.2. *Let (M^n, g) be a compact Riemannian manifold. Assume $g \in W^{s,p}(T^2M)$ with $sp > n$ and $s \geq 1$. Suppose $e \in \mathbb{R}$ and $q \in (1,\infty)$ are such that for balls $\Omega \subseteq \mathbb{R}^n$ or for $\Omega = \mathbb{R}^n$*

$$W^{s,p}(\Omega) \times W^{e,q}(\Omega) \hookrightarrow W^{e,q}(\Omega)\,.$$

Suppose $\{g_m\}$ is a sequence of smooth (C^∞) metrics on M such that $g_m \to g$ in $W^{s,p}(T^2M)$. Then

$$sharp_{g_m} \to sharp_g \quad in\ L(W^{e,q}(T^*M), W^{e,q}(TM))\,.$$

Proof.

$$\|\text{sharp}_{g_m} - \text{sharp}_g\|_{op} = \sup_{\|\omega\|_{e,q}\neq 0} \frac{\|(\text{sharp}_{g_m} - \text{sharp}_g)\omega\|_{W^{e,q}(TM)}}{\|\omega\|_{W^{e,q}(T^*M)}}\,.$$

Let $\Lambda = \{(U_\alpha, \varphi_\alpha, \rho_\alpha)\}_{\alpha=1}^N$ be a standard total trivialization atlas for TM and $\tilde{\Lambda} = \{(U_\alpha, \varphi_\alpha, \tilde{\rho}_\alpha)\}_{\alpha=1}^N$ be a standard total trivialization atlas for T^*M. Without loss of generality we may assume that each of Λ and $\tilde{\Lambda}$ is nice (or super nice) and GL compatible with itself. Let $\{\psi_\alpha\}_{\alpha=1}^N$ be a partition of unity subordinate to the open cover $\{U_\alpha\}_{\alpha=1}^N$.

Let $\tilde{\psi}_\alpha = \frac{\psi_\alpha^2}{\sum_{\beta=1}^N \psi_\beta^2}$. Note that $\frac{1}{\sum_{\beta=1}^N \psi_\beta^2} \circ \varphi_\alpha^{-1} \in BC^\infty(\varphi_\alpha(U_\alpha))$. We have

$$\|(\text{sharp}_{g_m} - \text{sharp}_g)\omega\|_{W^{e,q}(TM)} \simeq \sum_{\alpha=1}^N \sum_{i=1}^n \|\tilde{\psi}_\alpha(\rho_\alpha)^i(\text{sharp}_{g_m} - \text{sharp}_g)\omega \circ \varphi_\alpha^{-1}\|_{W^{e,q}(\varphi_\alpha(U_\alpha))}$$

$$\simeq \sum_{\alpha=1}^N \sum_{i=1}^n \|\tilde{\psi}_\alpha(g_m^{ij} - g^{ij})\omega_j \circ \varphi_\alpha^{-1}\|_{W^{e,q}(\varphi_\alpha(U_\alpha))}$$

$$\preceq \sum_{\alpha=1}^N \sum_{i=1}^n \sum_{j=1}^n \|\psi_\alpha(g_m^{ij} - g^{ij}) \circ \varphi_\alpha^{-1}\|_{W^{s,p}(\varphi_\alpha(U_\alpha))}\|\psi_\alpha\omega_j \circ \varphi_\alpha^{-1}\|_{W^{e,q}(\varphi_\alpha(U_\alpha))}$$

$$\preceq \|g_m^{-1} - g^{-1}\|_{W^{s,p}(T_2M)}\|\omega\|_{W^{e,q}(T^*M)} .$$

Now the claim follows from Theorem 4.3. □

If F is a general covariant k-tensor field ($k \geq 2$), we let $\text{sharp}_g F$ to be a $\binom{k-1}{1}$-tensor field defined by

$$\text{sharp}_g F(\omega, X_1, \cdots, X_{k-1}) := F(X_1, \cdots, X_{k-1}, \text{sharp}_g(\omega)) .$$

In any local coordinate chart

$$(\text{sharp}_g F)^j_{i_1\cdots i_{k-1}} = g^{jl} F_{i_1\cdots i_{k-1}l} .$$

The proof of the next two theorems is completely analogous to the proof of Theorems 5.1 and 5.2 and will be omitted.

Theorem 5.3. *Let (M^n, g) be a compact Riemannian manifold. Assume $g \in W^{s,p}(T^2M)$ with $sp > n$ and $s \geq 1$. Suppose $e \in \mathbb{R}$ and $q \in (1,\infty)$ are such that for balls $\Omega \subseteq \mathbb{R}^n$ or for $\Omega = \mathbb{R}^n$*

$$W^{s,p}(\Omega) \times W^{e,q}(\Omega) \hookrightarrow W^{e,q}(\Omega) .$$

Then $\text{sharp}_g : (C^\infty(M, T^kM), \|.\|_{e,q}) \to W^{e,q}(T_1^{k-1}M)$ is continuous and so it has a unique extension to a continuous operator $\text{sharp}_g : W^{e,q}(T^kM) \to W^{e,q}(T_1^{k-1}M)$.

Theorem 5.4. *Let (M^n, g) be a compact Riemannian manifold. Assume $g \in W^{s,p}(T^2M)$ with $sp > n$ and $s \geq 1$. Suppose $e \in \mathbb{R}$ and $p \in (1,\infty)$ are such that for balls $\Omega \subseteq \mathbb{R}^n$ or for $\Omega = \mathbb{R}^n$*

$$W^{s,p}(\Omega) \times W^{e,q}(\Omega) \hookrightarrow W^{e,q}(\Omega) .$$

Suppose $\{g_m\}$ is a sequence of smooth (C^∞) metrics on M such that $g_m \to g$ in $W^{s,p}(T^2M)$. Then

$$\text{sharp}_{g_m} \to \text{sharp}_g \quad \text{in } L(W^{e,q}(T^kM), W^{e,q}(T_1^{k-1}M)) .$$

6. Gradient with Rough Metric

Let M be a compact smooth manifold and let g be a Riemannian metric on M. Let $f : M \to \mathbb{R}$ be a scalar function. grad f is defined as $\text{sharp}_g(df)$. If $(U, (x^i))$ is any local coordinate chart, then

$$df = \frac{\partial f}{\partial x^i} dx^i, \quad \text{grad} f = [g^{ij}(\frac{\partial f}{\partial x^i})] \frac{\partial}{\partial x^j} .$$

Theorem 6.1. *Let (M^n, g) be a compact Riemannian manifold. Assume $g \in W^{s,p}(T^2M)$ with $sp > n$ and $s \geq 1$. Suppose $e \in \mathbb{R}$ and $p \in (1,\infty)$ are such that for balls $\Omega \subseteq \mathbb{R}^n$ or for $\Omega = \mathbb{R}^n$*

$$W^{s,p}(\Omega) \times W^{e,q}(\Omega) \hookrightarrow W^{e,q}(\Omega) .$$

Suppose $\{g_m\}$ is a sequence of smooth (C^∞) metrics on M such that $g_m \to g$ in $W^{s,p}(T^2M)$. Then

$$\textit{grad}_{g_m} \to \textit{grad}_g \quad \textit{in } L(W^{e+1,q}(M), W^{e,q}(TM))\,.$$

Proof. First note that, under the hypotheses of the theorem, grad_{g_m} and grad_g belong to $L(W^{e+1,q}(M), W^{e,q}(TM))$ (see Appendix A).

$$\begin{aligned}\|\text{grad}_{g_m} - \text{grad}_g\|_{L(W^{e+1,q},W^{e,q})} &= \|(\text{sharp}_{g_m} - \text{sharp}_g)\circ d\|_{L(W^{e+1,q},W^{e,q})}\\ &\preceq \|\text{sharp}_{g_m} - \text{sharp}_g\|_{L(W^{e,q}(T^*M),W^{e,q}(TM))}\|d\|_{L(W^{e+1,q}(M),W^{e,q}(T^*M))}\,.\end{aligned}$$

However, we have already proved that under the hypothesis of the theorem

$$\|\text{sharp}_{g_m} - \text{sharp}_g\|_{L(W^{e,q}(T^*M),W^{e,q}(TM))} \to 0\,.$$

Also, in Appendix A it is shown that $d : W^{e+1,q}(M) \to W^{e,q}(T^*M)$ is continuous. Therefore,

$$\|\text{grad}_{g_m} - \text{grad}_g\|_{L(W^{e+1,q},W^{e,q})} \to 0\,.$$

□

Alternatively, a rather special case of the above result can be proved using the technique introduced in [12] for $H^1(M)$. This will be the context of the following theorem.

Theorem 6.2. *Let (M^n, g) be a compact Riemannian manifold. Assume $g \in W^{s,p}(T^2M)$ with $sp > n$, $s \geq 1$. Suppose $\{g_m\}$ is a sequence of smooth (C^∞) metrics on M such that $g_m \to g$ in $W^{s,p}(T^2M)$. Then*

$$\textit{grad}_{g_m} \to \textit{grad}_g \quad \textit{in } L(W^{1,q}(M), L^q(TM))\,.$$

Proof. First note that since $sp > n$, $W^{s,p} \hookrightarrow L^\infty$ and therefore for all $1 < q < \infty$, we have

$$W^{s,p} \times L^q \hookrightarrow L^q\,.$$

Thus, this theorem is indeed a special case of the previous theorem. Denote the distortion tensor associated with g_m by A_m.

$$\begin{aligned}&\|\text{grad}_{g_m} - \text{grad}_g\|_{op}\\ &\overset{\text{Theorem 3.5}}{=} \sup\{|\langle Y, (\text{grad}_{g_m} - \text{grad}_g)u\rangle_{L^{q'}\times L^q}| : u \in C^\infty(M), Y \in C^\infty(TM),\ \|u\|_{1,q} = \|Y\|_{q'} = 1\}\\ &= \sup\{|\int_M g(Y, (A_m^{-1} - Id)\text{grad}_g u)\, dV_g| : u \in C^\infty(M), Y \in C^\infty(TM),\ \|u\|_{1,q} = \|Y\|_{q'} = 1\}\\ &\leq \sup\{\|A_m^{-1} - Id\|_\infty \int_M \|Y\|_g \|\text{grad}_g u\|_g\, dV_g : u \in C^\infty(M), Y \in C^\infty(TM),\ \|u\|_{1,q} = \|Y\|_{q'} = 1\}\,.\end{aligned}$$

Now, note that

$$\begin{aligned}\int_M \|Y\|_g \|\text{grad}_g u\|_g\, dV_g &\leq \big\|\|\text{grad}_g u\|_g\big\|_q \big\|\|Y\|_g\big\|_{q'}\\ &\preceq \|u\|_{1,q}\|Y\|_{q'} = 1\,.\end{aligned}$$

Therefore,

$$\|\text{grad}_{g_m} - \text{grad}_g\|_{op} \preceq \|A_m^{-1} - Id\|_\infty\,.$$

Finally, notice that by Theorem 4.2, $\|A_m^{-1} - Id\|_\infty \to 0$ as $m \to \infty$. □

7. Linear Connection with Rough Metric

Given a Riemannian manifold (M, g), we denote the corresponding Levi-Civita connection on TM by ∇_g.

Theorem 7.1. *Let (M^n, g) be a compact Riemannian manifold. Let $g \in W^{s,p}(T^2M)$ with $sp > n$ and $s \geq 1$. Suppose $e \in \mathbb{R}$ and $q \in (1, \infty)$ are such that*

$$W^{s-1,p}(\mathbb{R}^n) \times W^{e+1,q}(\mathbb{R}^n) \hookrightarrow W^{e,q}(\mathbb{R}^n)\,.$$

Also, let $X \in W^{\tilde{s},\tilde{p}}(TM)$ where $\tilde{s}$ and $\tilde{p}$ have the property that

$$W^{\tilde{s},\tilde{p}}(\mathbb{R}^n) \times W^{s-1,p}(\mathbb{R}^n) \times W^{e+1,q}(\mathbb{R}^n) \hookrightarrow W^{e,q}(\mathbb{R}^n)\,.$$

In particular, X can be any smooth vector field.

Suppose $\{g_m\}$ is a sequence of smooth (C^∞) metrics on M such that $g_m \to g$ in $W^{s,p}(T^2M)$. Then

$$(\nabla_{g_m})_X - (\nabla_g)_X \to 0 \quad \textit{in } L(W^{e+1,q}(T^k_l M), W^{e,q}(T^k_l M))\,.$$

Proof. In this proof we will not use the summation convention. Let $\Lambda = \{(U_\alpha, \varphi_\alpha, \rho_\alpha)\}_{\alpha=1}^N$ be a standard total trivialization atlas for $T^k_l(M) \to M$. Without loss of generality we may assume that Λ is super nice and GL compatible with itself. Let $\{\psi_\alpha\}_{\alpha=1}^N$ be a partition of unity subordinate to the open cover $\{U_\alpha\}_{\alpha=1}^N$. Let $\tilde{\psi}_\alpha = \frac{\psi_\alpha^3}{\sum_{\beta=1}^N \psi_\beta^3}$. Note that $\frac{1}{\sum_{\beta=1}^N \psi_\beta^3} \circ \varphi_\alpha^{-1} \in BC^\infty(\varphi_\alpha(U_\alpha))$. Using techniques discussed in Appendix A, one can show that under the hypotheses of the theorem, $(\nabla_{g_m})_X$ and $(\nabla_g)_X$ indeed belong to $L(W^{e+1,q}(T^k_l M), W^{e,q}(T^k_l M))$.

$$\|(\nabla_{g_m})_X - (\nabla_g)_X\|_{L(W^{e+1,q}(T^k_l M), W^{e,q}(T^k_l M))} = \sup_{F\neq 0, F\in C^\infty} \frac{\|(\nabla_{g_m})_X F - (\nabla_g)_X F\|_{e,q}}{\|F\|_{e+1,q}}\,.$$

We have (in what follows $\sum_{j_{\tilde{r}}, i_{\tilde{r}}}$ represents $\sum_{j_1=1}^n \cdots \sum_{j_l=1}^n \sum_{i_1=1}^n \cdots \sum_{i_k=1}^n$)

$$\|(\nabla_{g_m})_X F - (\nabla_g)_X F\|_{e,q} \simeq \sum_{\alpha=1}^N \sum_{j_{\tilde{r}}, i_{\tilde{r}}} \|\tilde{\psi}_\alpha[((\nabla_{g_m})_X F)^{j_1\cdots j_l}_{i_1\cdots i_k} - ((\nabla_g)_X F)^{j_1\cdots j_l}_{i_1\cdots i_k}] \circ \varphi_\alpha^{-1}\|_{W^{e,q}(\varphi_\alpha(U_\alpha))}\,.$$

Recall that on U_α,

$$(\nabla_{g_m})_X F = \sum_{r=1}^n X^r (\nabla_{g_m})_r F, \qquad (\nabla_g)_X F = \sum_{r=1}^n X^r (\nabla_g)_r F\,,$$

and

$$((\nabla_{g_m})_r F)^{j_1\cdots j_l}_{i_1\cdots i_k} \circ \varphi_\alpha^{-1} = \frac{\partial}{\partial x^r}(F^{j_1\cdots j_l}_{i_1\cdots i_k} \circ \varphi_\alpha^{-1})$$
$$+ \sum_{\hat{s}=1}^l \sum_{p=1}^n [F^{j_1\cdots p\cdots j_l}_{i_1\cdots i_k} \circ \varphi_\alpha^{-1}][(\Gamma_{g_m})^{j_{\hat{s}}}_{rp} \circ \varphi_\alpha^{-1}] - \sum_{\hat{s}=1}^k \sum_{p=1}^n [F^{j_1\cdots j_l}_{i_1\cdots p\cdots i_k} \circ \varphi_\alpha^{-1}][(\Gamma_{g_m})^p_{ri_{\hat{s}}} \circ \varphi_\alpha^{-1}]\,.$$

$$((\nabla_g)_r F)^{j_1\cdots j_l}_{i_1\cdots i_k} \circ \varphi_\alpha^{-1} = \frac{\partial}{\partial x^r}(F^{j_1\cdots j_l}_{i_1\cdots i_k} \circ \varphi_\alpha^{-1})$$
$$+ \sum_{\hat{s}=1}^l \sum_{p=1}^n [F^{j_1\cdots p\cdots j_l}_{i_1\cdots i_k} \circ \varphi_\alpha^{-1}][(\Gamma_g)^{j_{\hat{s}}}_{rp} \circ \varphi_\alpha^{-1}] - \sum_{\hat{s}=1}^k \sum_{p=1}^n [F^{j_1\cdots j_l}_{i_1\cdots p\cdots i_k} \circ \varphi_\alpha^{-1}][(\Gamma_g)^p_{ri_{\hat{s}}} \circ \varphi_\alpha^{-1}]\,.$$

(Here $F^{j_1\cdots p\cdots j_l}_{i_1\cdots i_k}$ represents $F^{j_1\cdots j_l}_{i_1\cdots i_k}$ with $j_{\hat{s}}$ replaced by p; similarly, $F^{j_1\cdots j_l}_{i_1\cdots p\cdots i_k}$ represents $F^{j_1\cdots j_l}_{i_1\cdots i_k}$ with $i_{\hat{s}}$ replaced by p.) Therefore,

$$[((\nabla_{g_m})_X F)^{j_1\cdots j_l}_{i_1\cdots i_k} - ((\nabla_g)_X F)^{j_1\cdots j_l}_{i_1\cdots i_k}]\circ\varphi_\alpha^{-1} =$$
$$\sum_{\hat{s}=1}^{l}\sum_{p=1}^{n}\sum_{r=1}^{n}(X^r\circ\varphi_\alpha^{-1})(F^{j_1\cdots p\cdots j_l}_{i_1\cdots i_k}\circ\varphi_\alpha^{-1})[(\Gamma_{g_m})^{j_{\hat{s}}}_{rp}\circ\varphi_\alpha^{-1}-(\Gamma_g)^{j_{\hat{s}}}_{rp}\circ\varphi_\alpha^{-1}]$$
$$-\sum_{\hat{s}=1}^{k}\sum_{p=1}^{n}\sum_{r=1}^{n}(X^r\circ\varphi_\alpha^{-1})(F^{j_1\cdots j_l}_{i_1\cdots p\cdots i_k}\circ\varphi_\alpha^{-1})[(\Gamma_{g_m})^{p}_{ri_{\hat{s}}}\circ\varphi_\alpha^{-1}-(\Gamma_g)^{p}_{ri_{\hat{s}}}\circ\varphi_\alpha^{-1}]\,.$$

Thus

$$\|(\nabla_{g_m})_X F-(\nabla_g)_X F\|_{e,q}\simeq$$
$$\sum_{\alpha=1}^{N}\sum_{j_{\tilde{r}},i_{\tilde{r}}}\Bigg\|\tilde{\psi}_\alpha\circ\varphi_\alpha^{-1}\Big[\sum_{\hat{s}=1}^{l}\sum_{p=1}^{n}\sum_{r=1}^{n}(X^r\circ\varphi_\alpha^{-1})(F^{j_1\cdots p\cdots j_l}_{i_1\cdots i_k}\circ\varphi_\alpha^{-1})[(\Gamma_{g_m})^{j_{\hat{s}}}_{rp}\circ\varphi_\alpha^{-1}-(\Gamma_g)^{j_{\hat{s}}}_{rp}\circ\varphi_\alpha^{-1}]$$
$$-\sum_{\hat{s}=1}^{k}\sum_{p=1}^{n}\sum_{r=1}^{n}(X^r\circ\varphi_\alpha^{-1})(F^{j_1\cdots j_l}_{i_1\cdots p\cdots i_k}\circ\varphi_\alpha^{-1})[(\Gamma_{g_m})^{p}_{ri_{\hat{s}}}\circ\varphi_\alpha^{-1}-(\Gamma_g)^{p}_{ri_{\hat{s}}}\circ\varphi_\alpha^{-1}]\Big]\Bigg\|_{W^{e,q}(\varphi_\alpha(U_\alpha))}$$

$$\preceq\sum_{\alpha=1}^{N}\sum_{j_{\tilde{r}},i_{\tilde{r}}}\sum_{\hat{s}=1}^{l}\sum_{p=1}^{n}\sum_{r=1}^{n}\Big[\|(\psi_\alpha\circ\varphi_\alpha^{-1})(X^r\circ\varphi_\alpha^{-1})\|_{\tilde{s},\tilde{p}}\|(\psi_\alpha\circ\varphi_\alpha^{-1})(F^{j_1\cdots p\cdots j_l}_{i_1\cdots i_k}\circ\varphi_\alpha^{-1})\|_{e+1,q}$$
$$\|(\psi_\alpha\circ\varphi_\alpha^{-1})[(\Gamma_{g_m})^{j_{\hat{s}}}_{rp}\circ\varphi_\alpha^{-1}-(\Gamma_g)^{j_{\hat{s}}}_{rp}\circ\varphi_\alpha^{-1}]\|_{s-1,p}\Big]$$
$$+\sum_{\alpha=1}^{N}\sum_{j_{\tilde{r}},i_{\tilde{r}}}\sum_{\hat{s}=1}^{k}\sum_{p=1}^{n}\sum_{r=1}^{n}\Big[\|(\psi_\alpha\circ\varphi_\alpha^{-1})(X^r\circ\varphi_\alpha^{-1})\|_{\tilde{s},\tilde{p}}\|(\psi_\alpha\circ\varphi_\alpha^{-1})(F^{j_1\cdots j_l}_{i_1\cdots p\cdots i_k}\circ\varphi_\alpha^{-1})\|_{e+1,q}$$
$$\|(\psi_\alpha\circ\varphi_\alpha^{-1})[(\Gamma_{g_m})^{p}_{ri_{\hat{s}}}\circ\varphi_\alpha^{-1}-(\Gamma_g)^{p}_{ri_{\hat{s}}}\circ\varphi_\alpha^{-1}]\|_{s-1,p}\Big]$$

$$\preceq\|X\|_{W^{\tilde{s},\tilde{p}}(TM)}\|F\|_{W^{e+1,q}(T^k_l(M))}\sum_{\alpha=1}^{N}\sum_{\hat{s}=1}^{l}\sum_{p=1}^{n}\sum_{r=1}^{n}\|(\psi_\alpha\circ\varphi_\alpha^{-1})[(\Gamma_{g_m})^{j_{\hat{s}}}_{rp}\circ\varphi_\alpha^{-1}-(\Gamma_g)^{j_{\hat{s}}}_{rp}\circ\varphi_\alpha^{-1}]\|_{W^{s-1,p}(\varphi_\alpha(U_\alpha))}$$
$$+\|X\|_{W^{\tilde{s},\tilde{p}}(TM)}\|F\|_{W^{e+1,q}(T^k_l(M))}\sum_{\alpha=1}^{N}\sum_{\hat{s}=1}^{k}\sum_{p=1}^{n}\sum_{r=1}^{n}\|(\psi_\alpha\circ\varphi_\alpha^{-1})[(\Gamma_{g_m})^{p}_{ri_{\hat{s}}}\circ\varphi_\alpha^{-1}-(\Gamma_g)^{p}_{ri_{\hat{s}}}\circ\varphi_\alpha^{-1}]\|_{W^{s-1,p}(\varphi_\alpha(U_\alpha))}\,.$$

Therefore,

$$\frac{\|(\nabla_{g_m})_X F-(\nabla_g)_X F\|_{e,q}}{\|F\|_{e+1,q}}\preceq\sum_{\alpha=1}^{N}\sum_{\hat{s}=1}^{l}\sum_{p=1}^{n}\sum_{r=1}^{n}\|(\psi_\alpha\circ\varphi_\alpha^{-1})[(\Gamma_{g_m})^{j_{\hat{s}}}_{rp}\circ\varphi_\alpha^{-1}-(\Gamma_g)^{j_{\hat{s}}}_{rp}\circ\varphi_\alpha^{-1}]\|_{W^{s-1,p}(\varphi_\alpha(U_\alpha))}$$
$$+\sum_{\alpha=1}^{N}\sum_{\hat{s}=1}^{k}\sum_{p=1}^{n}\sum_{r=1}^{n}\|(\psi_\alpha\circ\varphi_\alpha^{-1})[(\Gamma_{g_m})^{p}_{ri_{\hat{s}}}\circ\varphi_\alpha^{-1}-(\Gamma_g)^{p}_{ri_{\hat{s}}}\circ\varphi_\alpha^{-1}]\|_{W^{s-1,p}(\varphi_\alpha(U_\alpha))}\,.$$

Since $g_m\to g$ in $W^{s,p}$, it follows from Theorem 4.3 that the right hand side goes to zero as $m\to\infty$. □

8. Covariant Derivative with Rough Metric

Let $F\in\tau^k_l(M^n)$. The map

$$\nabla F:\tau^1(M)\times\cdots\times\tau^1(M)\times\chi(M)\times\cdots\times\chi(M)\to C^\infty(M)$$
$$(\omega^1,\ldots,\omega^l,Y_1,\ldots,Y_k,X)\mapsto(\nabla_X F)(\omega^1,\ldots,\omega^l,Y_1,\ldots,Y_k)\,.$$

is $C^\infty(M)$-multilinear and so it defines a $\binom{k+1}{l}$-tensor field. The tensor field ∇F is called the (total) covariant derivative of F. Note that in any local coordinates (in this section we do not use the summation convention)

$$(\nabla F)^{j_1\cdots j_l}_{i_1\cdots i_k r}\circ\varphi_\alpha^{-1}=(\nabla_r F)^{j_1\cdots j_l}_{i_1\cdots i_k}\circ\varphi_\alpha^{-1}$$

$$=\frac{\partial}{\partial x^r}(F^{j_1\cdots j_l}_{i_1\cdots i_k}\circ\varphi_\alpha^{-1})+\sum_{\hat{s}=1}^{l}\sum_{p=1}^{n}\left(F^{j_1\cdots p\cdots j_l}_{i_1\cdots i_k}\circ\varphi_\alpha^{-1}\right)\left(\Gamma^{j_{\hat{s}}}_{rp}\circ\varphi_\alpha^{-1}\right)-\sum_{\hat{s}=1}^{k}\sum_{p=1}^{n}\left(F^{j_1\cdots j_l}_{i_1\cdots p\cdots i_k}\circ\varphi_\alpha^{-1}\right)\left(\Gamma^{p}_{ri_{\hat{s}}}\circ\varphi_\alpha^{-1}\right).$$

Theorem 8.1. *Let (M^n,g) be a compact Riemannian manifold. Assume $g\in W^{s,p}(T^2M)$ with $sp>n$ and $s\geq 1$. Suppose $e\in\mathbb{R}$ and $q\in(1,\infty)$ are such that for balls $\Omega\subseteq\mathbb{R}^n$ or for $\Omega=\mathbb{R}^n$*

$$W^{s-1,p}(\Omega)\times W^{e+1,q}(\Omega)\hookrightarrow W^{e,q}(\Omega)\,.$$

In the case that the above multiplication property holds only for balls $\Omega\subseteq\mathbb{R}^n$ and not $\mathbb{R}^n$ itself, further assume that e and q are such that $\frac{\partial}{\partial x^j}:W^{e+1,q}(\Omega)\to W^{e,q}(\Omega)$ $(1\leq j\leq n)$ is continuous (see Theorem 3.9).

Suppose $\{g_m\}$ is a sequence of smooth (C^∞) metrics on M such that $g_m\to g$ in $W^{s,p}(T^2M)$. Then

$$\nabla_{g_m}\to\nabla_g\quad\text{in } L(W^{e+1,q}(T^k_lM),W^{e,q}(T^{k+1}_lM))\,.$$

Proof. Let $\Lambda=\{(U_\alpha,\varphi_\alpha,\rho_\alpha)\}_{\alpha=1}^N$ and $\tilde{\Lambda}=\{(U_\alpha,\varphi_\alpha,\tilde{\rho}_\alpha)\}_{\alpha=1}^N$ be standard total trivialization atlases for $T^k_l(M)\to M$ and $T^{k+1}_l(M)\to M$, respectively. Without loss of generality, we may assume that each of Λ and $\tilde{\Lambda}$ is nice (or super nice) and GL compatible with itself. Let $\{\psi_\alpha\}_{\alpha=1}^N$ be a partition of unity subordinate to the open cover $\{U_\alpha\}_{\alpha=1}^N$. Let $\tilde{\psi}_\alpha=\frac{\psi_\alpha^2}{\sum_{\beta=1}^N\psi_\beta^2}$. Note that $\frac{1}{\sum_{\beta=1}^N\psi_\beta^2}\circ\varphi_\alpha^{-1}\in BC^\infty(\varphi_\alpha(U_\alpha))$. Also under the hypotheses of the theorem, ∇_{g_m} and ∇_g belong to $L(W^{e+1,q}(T^k_lM),W^{e,q}(T^{k+1}_lM))$ (see Example 5 in Appendix A).

$$\|\nabla_{g_m}-\nabla_g\|_{L(W^{e+1,q}(T^k_lM),W^{e,q}(T^{k+1}_lM))}=\sup_{F\neq0,F\in C^\infty}\frac{\|\nabla_{g_m}F-\nabla_gF\|_{e,q}}{\|F\|_{e+1,q}}\,.$$

We have (in what follows $\sum_{j_{\tilde{r}},i_{\tilde{r}},r}$ represents $\sum_{j_1=1}^n\cdots\sum_{j_l=1}^n\sum_{i_1=1}^n\cdots\sum_{i_k=1}^n\sum_{r=1}^n$)

$$\|\nabla_{g_m}F-\nabla_gF\|_{e,q}\simeq\sum_{\alpha=1}^N\sum_{j_{\tilde{r}},i_{\tilde{r}},r}\|\tilde{\psi}_\alpha[(\nabla_{g_m}F)^{j_1\cdots j_l}_{i_1\cdots i_k r}-(\nabla_gF)^{j_1\cdots j_l}_{i_1\cdots i_k r}]\circ\varphi_\alpha^{-1}\|_{W^{e,q}(\varphi_\alpha(U_\alpha))}$$

$$=\sum_{\alpha=1}^N\sum_{j_{\tilde{r}},i_{\tilde{r}},r}\|\tilde{\psi}_\alpha[((\nabla_{g_m})_rF)^{j_1\cdots j_l}_{i_1\cdots i_k}-((\nabla_g)_rF)^{j_1\cdots j_l}_{i_1\cdots i_k}]\circ\varphi_\alpha^{-1}\|_{W^{e,q}(\varphi_\alpha(U_\alpha))}\,.$$

The exact same procedure as the one given in the proof of Theorem 7.1 shows that the above expression is bounded by a constant times

$$\|F\|_{e+1,q}\Big[\sum_{\alpha=1}^N\sum_{\hat{s}=1}^l\sum_{p=1}^n\sum_{r=1}^n\|(\psi_\alpha\circ\varphi_\alpha^{-1})[(\Gamma_{g_m})^{j_{\hat{s}}}_{rp}\circ\varphi_\alpha^{-1}-(\Gamma_g)^{j_{\hat{s}}}_{rp}\circ\varphi_\alpha^{-1}]\|_{W^{s-1,p}(\varphi_\alpha(U_\alpha))}$$

$$+\sum_{\alpha=1}^N\sum_{\hat{s}=1}^k\sum_{p=1}^n\sum_{r=1}^n\|(\psi_\alpha\circ\varphi_\alpha^{-1})[(\Gamma_{g_m})^{p}_{ri_{\hat{s}}}\circ\varphi_\alpha^{-1}-(\Gamma_g)^{p}_{ri_{\hat{s}}}\circ\varphi_\alpha^{-1}]\|_{W^{s-1,p}(\varphi_\alpha(U_\alpha))}\Big]\,.$$

Since $g_m\to g$ in $W^{s,p}$, it follows from Theorem 4.3 that the right hand side divided by $\|F\|_{e+1,q}$ goes to zero as $m\to\infty$. □

9. Continuity of Trace

It is well known that we can associate with any $\binom{1}{1}$ tensor field a corresponding field of endomorphisms of tangent spaces. If F is a $\binom{1}{1}$ tensor field, then the trace of F at each point $p \in M$ is defined as the trace of the corresponding endomorphism of T_pM. So, tr F will be a scalar field on M. More generally, let F be a $\binom{k}{l}$-tensor field where $k, l \geq 1$. We can define the trace of F with respect to the pair (r, s) $(1 \leq r \leq l, 1 \leq s \leq k)$ as follows: $\operatorname{tr} F$ is a $\binom{k-1}{l-1}$-tensor field defined by

$$(\operatorname{tr} F)(\omega^1, \ldots, \omega^{r-1}, \omega^{r+1}, \ldots, \omega^l, X_1, \ldots, X_{s-1}, X_{s+1}, \ldots, X_k) := \operatorname{tr} G\,,$$

where $G \in T^1_1(V)$ is given by

$$G(\omega, X) := F(\omega^1, \ldots, \omega^{r-1}, \omega, \omega^{r+1}, \ldots, \omega^l, X_1, \ldots, X_{s-1}, X, X_{s+1}, \ldots, X_k).$$

In this section, in computing trace we assume $(r, s) = (l, k)$. With respect to any local coordinate chart we have

$$(\operatorname{tr} F)^{j_1 \ldots j_{l-1}}_{i_1 \ldots i_{k-1}} = F^{j_1 \ldots j_{l-1} m}_{i_1 \ldots i_{k-1} m}.$$

Theorem 9.1. *Let M^n be a compact smooth manifold. Let $e \in \mathbb{R}$ and $q \in (1, \infty)$. Suppose $k, l \geq 1$. Then tr : $(C^\infty(M, T^k_l(M)), \|.\|_{e,q}) \to W^{e,q}(T^{k-1}_{l-1}(M))$ is continuous and so it has a unique extension to a continuous operator tr : $W^{e,q}(T^k_l(M)) \to W^{e,q}(T^{k-1}_{l-1}(M))$.*

Proof. Let $\{(U_\alpha, \varphi_\alpha, \rho_\alpha)\}_{\alpha=1}^N$ be a standard total trivialization atlas for $T^k_l(M) \to M$ that is GL compatible with itself. Let $\{\psi_\alpha\}_{\alpha=1}^N$ be a partition of unity subordinate to the open cover $\{U_\alpha\}_{\alpha=1}^N$. Note that $T^k_l(M)$ is a bundle of rank n^{k+l}. So for each α, ρ_α has n^{k+l} components which we denote by $(\rho_\alpha)^{j_1 \cdots j_l}_{i_1 \cdots i_k}$. For all $F \in \Gamma(M, T^k_l(M))$, we have

$$(\rho_\alpha)^{j_1 \cdots j_l}_{i_1 \cdots i_k}(\psi_\alpha F) = \psi_\alpha (F_\alpha)^{j_1 \cdots j_l}_{i_1 \cdots i_k}\,,$$

where $F = (F_\alpha)^{j_1 \cdots j_l}_{i_1 \cdots i_k} \partial_{j_1} \otimes \cdots \otimes \partial_{j_l} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_k}$ on the coordinate chart $(U_\alpha, \varphi_\alpha)$.

Therefore, (in what follows $\sum_{j_r, i_r}$ represents $\sum_{j_1=1}^n \cdots \sum_{j_{l-1}=1}^n \sum_{i_1=1}^n \cdots \sum_{i_{k-1}=1}^n$)

$$\|\operatorname{tr} F\|^q_{W^{e,q}(T^{k-1}_{l-1}(M))} \simeq \sum_{\alpha=1}^N \sum_{j_r, i_r} \|(\rho_\alpha)^{j_1 \cdots j_{l-1}}_{i_1 \cdots i_{k-1}} \circ (\psi_\alpha \operatorname{tr} F) \circ \varphi_\alpha^{-1}\|^q_{W^{e,q}(\varphi_\alpha(U_\alpha))}$$

$$= \sum_{\alpha=1}^N \sum_{j_r, i_r} \|(\psi_\alpha)((\operatorname{tr} F)_\alpha)^{j_1 \cdots j_{l-1}}_{i_1 \cdots i_{k-1}} \circ \varphi_\alpha^{-1}\|^q_{W^{e,q}(\varphi_\alpha(U_\alpha))}$$

$$= \sum_{\alpha=1}^N \sum_{j_r, i_r} \|(\psi_\alpha)(F_\alpha)^{j_1 \cdots j_{l-1} m}_{i_1 \cdots i_{k-1} m} \circ \varphi_\alpha^{-1}\|^q_{W^{e,q}(\varphi_\alpha(U_\alpha))}$$

$$\leq \sum_{\alpha=1}^N \sum_{j_r, i_r} \|(\psi_\alpha)(F_\alpha)^{j_1 \cdots j_{l-1} j_l}_{i_1 \cdots i_{k-1} i_k} \circ \varphi_\alpha^{-1}\|^q_{W^{e,q}(\varphi_\alpha(U_\alpha))} \quad \text{(this sum has more terms comparing to the last)}$$

$$= \sum_{\alpha=1}^N \sum_{j_r, i_r} \|(\rho_\alpha)^{j_1 \cdots j_l}_{i_1 \cdots i_k} \circ (\psi_\alpha F) \circ \varphi_\alpha^{-1}\|^q_{W^{e,q}(\varphi_\alpha(U_\alpha))}$$

$$= \|F\|^q_{W^{e,q}(T^k_l(M))}\,.$$

□

Note that in the above proof the trace was computed on the last pair of indices. Of course, clearly the same procedure shows that taking trace on any pair of indices is continuous.

10. Divergence with Rough Metric

We begin with studying the divergence of a vector field. Then we will consider the divergence of more general tensor fields.

Theorem 10.1. *Let (M^n, g) be a compact Riemannian manifold. Assume $g \in W^{s,p}(T^2M)$ with $sp > n$ and $s \geq 1$. Suppose $e \in \mathbb{R}$ and $q \in (1,\infty)$ are such that for balls $\Omega \subseteq \mathbb{R}^n$ or for $\Omega = \mathbb{R}^n$*

$$W^{s,p}(\Omega) \times W^{e+1,q}(\Omega) \hookrightarrow W^{e+1,q}(\Omega)\,,$$
$$W^{s,p}(\Omega) \times W^{e,q}(\Omega) \hookrightarrow W^{e,q}(\Omega)\,,$$
$$W^{s-1,p}(\Omega) \times W^{e+1,q}(\Omega) \hookrightarrow W^{e,q}(\Omega)\,.$$

In the case that the above multiplication property holds only for balls $\Omega \subseteq \mathbb{R}^n$ and not $\mathbb{R}^n$ itself, further assume that e and q are such that $\frac{\partial}{\partial x^j} : W^{e+1,q}(\Omega) \to W^{e,q}(\Omega)$ ($1 \leq j \leq n$) is continuous.

Suppose $\{g_m\}$ is a sequence of smooth (C^∞) metrics on M such that $g_m \to g$ in $W^{s,p}(T^2M)$. Then

$$div_{g_m} \to div_g \quad in\ L(W^{e+1,q}(TM), W^{e,q}(M))\,.$$

Proof. Let $\Lambda = \{(U_\alpha, \varphi_\alpha, \rho_\alpha)\}_{\alpha=1}^N$ be a standard total trivialization atlas for TM. Without loss of generality we may assume that Λ is nice (or super nice) and GL compatible with itself. Let $\{\psi_\alpha\}_{\alpha=1}^N$ be a partition of unity subordinate to the open cover $\{U_\alpha\}_{\alpha=1}^N$. Let $\tilde{\psi}_\alpha = \frac{\psi_\alpha^2}{\sum_{\beta=1}^N \psi_\beta^2}$. Note that $\frac{1}{\sum_{\beta=1}^N \psi_\beta^2} \circ \varphi_\alpha^{-1} \in BC^\infty(\varphi_\alpha(U_\alpha))$. Also, div_{g_m} and div_g belong to $L(W^{e+1,q}(TM), W^{e,q}(M))$ (see Example 3 in Appendix A). We have

$$\|\mathrm{div}_{g_m} - \mathrm{div}_g\|_{op} = \sup_{\|X\|_{e+1,q}=1, X \in C^\infty} \|(\mathrm{div}_{g_m} - \mathrm{div}_g)X\|_{W^{e,q}(M)}\,.$$

Note that

$$\|(\mathrm{div}_{g_m} - \mathrm{div}_g)X\|_{W^{e,q}(M)} \simeq \sum_{\alpha=1}^N \|\tilde{\psi}_\alpha\big((\mathrm{div}_{g_m} - \mathrm{div}_g)X\big) \circ \varphi_\alpha^{-1}\|_{W^{e,q}(\varphi_\alpha(U_\alpha))}$$
$$= \sum_{\alpha=1}^N \|(\tilde{\psi}_\alpha \circ \varphi_\alpha^{-1})\big((\mathrm{div}_{g_m} X \circ \varphi_\alpha^{-1}) - (\mathrm{div}_g X \circ \varphi_\alpha^{-1})\big)\|_{W^{e,q}(\varphi_\alpha(U_\alpha))}\,.$$

Recall that (in what follows we will not use the summation convention)

$$\mathrm{div}_g X \circ \varphi_\alpha^{-1} = \sum_{j=1}^n \frac{1}{\sqrt{\det g \circ \varphi_\alpha^{-1}}}\big(\frac{\partial}{\partial x^j}(\sqrt{\det g} \circ \varphi_\alpha^{-1})\big)(X^j \circ \varphi_\alpha^{-1}) + \frac{\partial}{\partial x^j}(X^j \circ \varphi_\alpha^{-1})\,,$$

$$\mathrm{div}_{g_m} X \circ \varphi_\alpha^{-1} = \sum_{j=1}^n \frac{1}{\sqrt{\det g_m \circ \varphi_\alpha^{-1}}}\big(\frac{\partial}{\partial x^j}(\sqrt{\det g_m} \circ \varphi_\alpha^{-1})\big)(X^j \circ \varphi_\alpha^{-1}) + \frac{\partial}{\partial x^j}(X^j \circ \varphi_\alpha^{-1})\,.$$

Therefore,

$$\mathrm{div}_{g_m} X \circ \varphi_\alpha^{-1} - \mathrm{div}_g X \circ \varphi_\alpha^{-1} =$$
$$\sum_{j=1}^n \big[\frac{1}{\sqrt{\det g_m \circ \varphi_\alpha^{-1}}}(\frac{\partial}{\partial x^j}(\sqrt{\det g_m} \circ \varphi_\alpha^{-1})) - \frac{1}{\sqrt{\det g \circ \varphi_\alpha^{-1}}}(\frac{\partial}{\partial x^j}(\sqrt{\det g} \circ \varphi_\alpha^{-1}))\big](X^j \circ \varphi_\alpha^{-1})\,.$$

Let

$$B_m = \frac{1}{\sqrt{\det g_m \circ \varphi_\alpha^{-1}}} (\frac{\partial}{\partial x^j}(\sqrt{\det g_m} \circ \varphi_\alpha^{-1})) ,$$
$$B = \frac{1}{\sqrt{\det g \circ \varphi_\alpha^{-1}}} (\frac{\partial}{\partial x^j}(\sqrt{\det g} \circ \varphi_\alpha^{-1})) .$$

Since $s > \frac{n}{p}$, $W^{s,p} \times W^{s-1,p} \hookrightarrow W^{s-1,p}$. Considering this, it follows from Theorem 3.14, Theorem 3.11, and Theorem 3.24 that $B_m - B \in W^{s-1,p}_{loc}$. Also, note that $X \in W^{e+1,q}$. So,

$$(\psi_\alpha \circ \varphi_\alpha^{-1})(B_m - B) \in W^{s-1,p}_{loc}(\varphi_\alpha(U_\alpha)) ,$$
$$(\psi_\alpha \circ \varphi_\alpha^{-1})(X^j \circ \varphi_\alpha^{-1}) \in W^{e+1,q}_{loc}(\varphi_\alpha(U_\alpha)) .$$

By assumption $W^{s-1,p} \times W^{e+1,q} \hookrightarrow W^{e,q}$. Consequently, we can write

$$\|(\mathrm{div}_{g_m} - \mathrm{div}_g) X\|_{W^{e,q}(M)} \preceq \sum_{\alpha=1}^{N}\sum_{j=1}^{n} \|(\psi_\alpha \circ \varphi_\alpha^{-1})(B_m - B)(\psi_\alpha \circ \varphi_\alpha^{-1})(X^j \circ \varphi_\alpha^{-1})\|_{W^{e,q}(\varphi_\alpha(U_\alpha))}$$
$$\preceq \sum_{\alpha=1}^{N}\sum_{j=1}^{n} \|(\psi_\alpha \circ \varphi_\alpha^{-1})(B_m - B)\|_{W^{s-1,p}(\varphi_\alpha(U_\alpha))} \|(\psi_\alpha \circ \varphi_\alpha^{-1})(X^j \circ \varphi_\alpha^{-1})\|_{W^{e+1,q}(\varphi_\alpha(U_\alpha))}$$
$$\simeq \|(\psi_\alpha \circ \varphi_\alpha^{-1})(B_m - B)\|_{W^{s-1,p}(\varphi_\alpha(U_\alpha))} \|X\|_{W^{e+1,q}(TM)} .$$

By assumption $g_m \to g$ in $W^{s,p}$. Therefore, $(g_m)_\alpha \to g_\alpha$ in $W^{s,p}_{loc}$. Consequently, $B_m \to B$ in $W^{s-1,p}_{loc}$. Thus $(\psi_\alpha \circ \varphi_\alpha^{-1})B_m \to (\psi_\alpha \circ \varphi_\alpha^{-1})B$ in $W^{s-1,p}$. □

Theorem 10.2. *Let (M^n, g) be a compact Riemannian manifold. Assume $g \in W^{s,p}(T^2M)$ with $sp > n$ and $s \geq 1$. Suppose $e \in \mathbb{R}$ and $q \in (1, \infty)$ are such that for balls $\Omega \subseteq \mathbb{R}^n$ or for $\Omega = \mathbb{R}^n$*

$$W^{s-1,p}(\Omega) \times W^{e+1,q}(\Omega) \hookrightarrow W^{e,q}(\Omega) .$$

In the case that the above multiplication property holds only for balls $\Omega \subseteq \mathbb{R}^n$ and not $\mathbb{R}^n$ itself, further assume that e and q are such that $\frac{\partial}{\partial x^j} : W^{e+1,q}(\Omega) \to W^{e,q}(\Omega)$ $(1 \leq j \leq n)$ is continuous.

Suppose $\{g_m\}$ is a sequence of smooth (C^∞) metrics on M such that $g_m \to g$ in $W^{s,p}(T^2M)$. Assume $k \geq 0$ and $l \geq 1$. Then

$$div_{g_m} \to div_g \quad in\ L(W^{e+1,q}(T^k_l M), W^{e,q}(T^k_{l-1} M)) .$$

Proof. The divergence of a tensor field F is defined as the trace of the total covariant derivative of F:

$$\mathrm{div} F = \mathrm{tr}(\nabla F) .$$

By Theorem 8.1,

$$\nabla_{g_m} \to \nabla_g \qquad \text{in } L(W^{e+1,q}(T^k_l M), W^{e,q}(T^{k+1}_l M)) .$$

Also, by Theorem 9.1, tr : $W^{e,q}(T^{k+1}_l M) \to W^{e,q}(T^k_{l-1} M)$ is a linear continuous operator. Therefore, by Theorem 3.4,

$$\mathrm{tr} \circ \nabla_{g_m} \to \mathrm{tr} \circ \nabla_g \qquad \text{in } L(W^{e+1,q}(T^k_l M), W^{e,q}(T^k_{l-1} M)) .$$

□

For a general $\binom{k}{0}$-tensor field F $(k \geq 1)$, ∇F is a $\binom{k+1}{0}$-tensor field and sharp(∇F) is a $\binom{k}{1}$- tensor field. Divergence of F is the $\binom{k-1}{0}$-tensor field defined by

$$\mathrm{div} F := \mathrm{tr}(\mathrm{sharp}(\nabla F)) .$$

Theorem 10.3. *Let (M^n, g) be a compact Riemannian manifold. Assume $g \in W^{s,p}(T^2M)$ with $sp > n$ and $s \geq 1$. Suppose $\{g_m\}$ is a sequence of smooth (C^∞) metrics on M such that $g_m \to g$ in $W^{s,p}(T^2M)$. Suppose $e \in \mathbb{R}$ and $q \in (1,\infty)$ are such that either*

$$W^{s-1,p}(\mathbb{R}^n) \times W^{e+1,q}(\mathbb{R}^n) \hookrightarrow W^{e,q}(\mathbb{R}^n)\,,$$
$$W^{s,p}(\mathbb{R}^n) \times W^{e,q}(\mathbb{R}^n) \hookrightarrow W^{e,q}(\mathbb{R}^n)\,,$$

or for balls $\Omega \subseteq \mathbb{R}^n$, $\frac{\partial}{\partial x^j} : W^{e+1,q}(\Omega) \to W^{e,q}(\Omega)$ $(1 \leq j \leq n)$ is continuous and

$$W^{s-1,p}(\Omega) \times W^{e+1,q}(\Omega) \hookrightarrow W^{e,q}(\Omega)\,,$$
$$W^{s,p}(\Omega) \times W^{e,q}(\Omega) \hookrightarrow W^{e,q}(\Omega)\,.$$

Assume $k \geq 1$. Then

$$div_{g_m} \to div_g \quad \text{in } L(W^{e+1,q}(T^kM), W^{e,q}(T^{k-1}M))\,.$$

Proof. By Theorem 8.1,

$$\nabla_{g_m} \to \nabla_g \qquad \text{in } L(W^{e+1,q}(T^kM), W^{e,q}(T^{k+1}M))\,.$$

By Theorem 5.4,

$$\text{sharp}_{g_m} \to \text{sharp}_g \quad \text{in } L(W^{e,q}(T^{k+1}M), W^{e,q}(T^k_1M))\,.$$

Also, by Theorem 9.1, $\text{tr} : W^{e,q}(T^k_1M) \to W^{e,q}(T^{k-1}M)$ is a linear continuous operator $(\text{tr} \in L(W^{e,q}(T^k_1M), W^{e,q}(T^{k-1}M)))$. It follows from Theorem 3.4 that

$$\text{tr} \circ \text{sharp}_{g_m} \circ \nabla_{g_m} \to \text{tr} \circ \text{sharp}_g \circ \nabla_g \qquad \text{in } L(W^{e+1,q}(T^kM), W^{e,q}(T^{k-1}M))\,.$$

□

11. Laplacian with Rough Metric

Theorem 11.1. *Let (M^n, g) be a compact Riemannian manifold. Assume $g \in W^{s,p}(T^2M)$ with $sp > n$ and $s \geq 1$. Suppose $\{g_m\}$ is a sequence of smooth (C^∞) metrics on M such that $g_m \to g$ in $W^{s,p}(T^2M)$. Suppose $e \in \mathbb{R}$ and $q \in (1,\infty)$ are such that either*

$$W^{s,p}(\mathbb{R}^n) \times W^{e,q}(\mathbb{R}^n) \hookrightarrow W^{e,q}(\mathbb{R}^n)\,,$$
$$W^{s,p}(\mathbb{R}^n) \times W^{e-1,q}(\mathbb{R}^n) \hookrightarrow W^{e-1,q}(\mathbb{R}^n)\,,$$
$$W^{s-1,p}(\mathbb{R}^n) \times W^{e,q}(\mathbb{R}^n) \hookrightarrow W^{e-1,q}(\mathbb{R}^n)\,,$$

or for balls $\Omega \subseteq \mathbb{R}^n$, $\frac{\partial}{\partial x^j} : W^{e,q}(\Omega) \to W^{e-1,q}(\Omega)$ $(1 \leq j \leq n)$ is continuous and

$$W^{s,p}(\Omega) \times W^{e,q}(\Omega) \hookrightarrow W^{e,q}(\Omega)\,,$$
$$W^{s,p}(\Omega) \times W^{e-1,q}(\Omega) \hookrightarrow W^{e-1,q}(\Omega)\,,$$
$$W^{s-1,p}(\Omega) \times W^{e,q}(\Omega) \hookrightarrow W^{e-1,q}(\Omega)\,.$$

Then

$$\Delta_{g_m} \to \Delta_g \quad \text{in } L(W^{e+1,q}(M), W^{e-1,q}(M))\,.$$

Proof. Note that $\Delta = \text{div} \circ \text{grad}$. By Theorem 6.1,

$$\text{grad}_{g_m} \to \text{grad}_g \qquad \text{in } L(W^{e+1,q}(M) \to W^{e,q}(TM))\,.$$

Also, by Theorem 10.1,

$$\text{div}_{g_m} \to \text{div}_g \qquad \text{in } L(W^{e,q}(TM) \to W^{e-1,q}(M))\,.$$

Therefore, it follows from Theorem 3.4 that

$$\text{div}_{g_m} \circ \text{grad}_{g_m} \to \text{div}_g \circ \text{grad}_g \qquad \text{in } L(W^{e+1,q}(M) \to W^{e-1,q}(M))\,.$$

□

As an alternative, for a certain range of Sobolev spaces, we may use the technique employed in [12] to prove the following result.

Theorem 11.2. *Let (M^n, g) be a compact Riemannian manifold. Assume $g \in W^{s,p}(T^2M)$ with $sp > n$, $s \geq 1$. Further assume that*

$$W^{s,p}(\mathbb{R}^n) \times W^{-1,p}(\mathbb{R}^n) \hookrightarrow W^{-1,p}(\mathbb{R}^n)\,.$$

Suppose $\{g_m\}$ is a sequence of smooth (C^∞) metrics on M such that $g_m \to g$ in $W^{s,p}(T^2M)$. Let A_m denote the metric distortion tensor associated with g_m and further assume det $A_m \to 1$ uniformly. Then

$$\Delta_{g_m} \to \Delta_g \quad \text{in } L(W^{1,p}(M), W^{-1,p}(M))\,.$$

Proof. First note that since $sp > n$, we have $W^{s,p}(\mathbb{R}^n) \times W^{0,p}(\mathbb{R}^n) \hookrightarrow W^{0,p}(\mathbb{R}^n)$. This together with the assumption that $W^{s,p}(\mathbb{R}^n) \times W^{-1,p}(\mathbb{R}^n) \hookrightarrow W^{-1,p}(\mathbb{R}^n)$ ensures that $\Delta_g = \text{div}_g \circ \text{grad}_g$ is a well-defined continuous operator from $W^{1,p}(M)$ to $W^{-1,p}(M)$ (see Appendix A).
By Theorem 4.2 we have

$$\|\sqrt{\det A_m}A_m^{-1} - Id\|_\infty \to 0, \quad A_m^{-1}\text{grad}_g = \text{grad}_{g_m}, \quad dV_{g_m} = \sqrt{\det A_m}dV_g\,.$$

So, it is enough to show that

$$\|\sqrt{\det A_m}A_m^{-1} - Id\|_\infty \to 0 \Longrightarrow \|\Delta_{g_m} - \Delta_g\|_{L(W^{1,p}(M),W^{-1,p}(M))} \to 0\,.$$

For all u and v in $C^\infty(M)$,

$$\begin{aligned}
\langle v, \Delta_{g_m}u\rangle_{W^{1,p'}\times W^{-1,p}} &= \int_M (\Delta_{g_m}u)\, v\, dV_{g_m} \quad \text{(see Theorem 3.26)}\\
&= -\int_M g_m(\text{grad}_{g_m}u, \text{grad}_{g_m}v)\, dV_{g_m} \quad \text{(integration by parts)}\\
&= -\int_M g(A_m\text{grad}_{g_m}u, \text{grad}_{g_m}v)\sqrt{\det A_m}\, dV_g\\
&= -\int_M g(A_mA_m^{-1}\text{grad}_g u, A_m^{-1}\text{grad}_g v)\sqrt{\det A_m}\, dV_g\\
&= -\int_M g(A_m^{-1}\text{grad}_g u, \text{grad}_g v)\sqrt{\det A_m}\, dV_g\,.
\end{aligned}$$

In the last equality we used the fact that A_m and A_m^{-1} are symmetric. Also,

$$\langle v, \Delta_g u\rangle_{W^{1,p'}\times W^{-1,p}} = \int_M (\Delta_g u)\, v\, dV_g = -\int_M g(\text{grad}_g u, \text{grad}_g v)\, dV_g\,.$$

Therefore,

$$\begin{aligned}
&\|\Delta_{g_m} - \Delta_g\|_{op} \overset{\text{Theorem 3.5}}{=} \sup\{|\langle v, (\Delta_{g_m} - \Delta_g)u\rangle| : u, v \in C^\infty(M),\ \|u\|_{1,p} = \|v\|_{1,p'} = 1\}\\
&= \sup\{|-\int_M g((\sqrt{\det A_m}A_m^{-1} - Id)\text{grad}_g u, \text{grad}_g v)\, dV_g| : u, v \in C^\infty(M),\ \|u\|_{1,p} = \|v\|_{1,p'} = 1\}\\
&\leq \sup\{\|\sqrt{\det A_m}A_m^{-1} - Id\|_\infty \int_M \|\text{grad}_g u\|_g\|\text{grad}_g v\|_g\, dV_g : u, v \in C^\infty(M),\ \|u\|_{1,p} = \|v\|_{1,p'} = 1\}\,.
\end{aligned}$$

Now note that,

$$\begin{aligned}
\int_M \|\text{grad}_g u\|_g\|\text{grad}_g v\|_g\, dV_g &\leq \Big(\big\|\|\text{grad}_g u\|_g\big\|_p\Big)\Big(\big\|\|\text{grad}_g v\|_g\big\|_{p'}\Big)\\
&= \|\text{grad}_g u\|_p\|\text{grad}_g v\|_{p'} \preceq \|u\|_{1,p}\|v\|_{1,p'} = 1\,.
\end{aligned}$$

Hence

$$\|\Delta_{g_m} - \Delta_g\|_{op} \preceq \|\sqrt{\det A_m} A_m^{-1} - Id\|_\infty .$$

□

12. Conformal Killing Operator with Rough Metric

Suppose (M, g) is a Riemannian manifold and ∇ is the corresponding Levi-Civita connection. For all vector fields $X, Y, Z \in C^\infty(TM)$ we have

$$\begin{aligned}(L_X g)(Y, Z) &= X(g(Y, Z)) - g([X, Y], Z) - g(Y, [X, Z]) \\ &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - g([X, Y], Z) - g(Y, [X, Z]) \\ &= g(\nabla_X Y - [X, Y], Z) + g(Y, \nabla_X Z - [X, Z]) \\ &= g(\nabla_Y X, Z) + g(Y, \nabla_Z X) .\end{aligned}$$

Here L_X denotes the Lie derivative with respect to the vector field X. Therefore, with respect to any local coordinate chart we have

$$L_X g_{ij} = \nabla_i X_j + \nabla_j X_i .$$

It follows that $\mathrm{tr}(L_X g) = 2\mathrm{div} X$. Therefore we can decompose $L_X g$ into the pure trace part and the trace-free part as follows:

$$L_X g = \underbrace{\left[\frac{1}{n}(2\mathrm{div} X) g\right]}_{\text{pure trace}} + \underbrace{\left[L_X g - \frac{1}{n}(2\mathrm{div} X) g\right]}_{\text{trace-free}} .$$

The *conformal Killing operator*, $\mathcal{L}$, is defined as follows:

$$\mathcal{L} X := \text{the trace-free part of } L_X g .$$

That is, with respect to any local chart (U, φ)

$$(\mathcal{L} X)_{ij} = \nabla_i X_j + \nabla_j X_i - \frac{2}{n}(\mathrm{div} X) g_{ij} .$$

Note that

$$\nabla_i X = (\partial_i X^l + X^k \Gamma^l_{ik}) \partial_l .$$

Therefore,

$$\begin{aligned}[\nabla_i X]_j + [\nabla_j X]_i &= g_{jl}[\nabla_i X]^l + g_{il}[\nabla_j X]^l \\ &= g_{jl}[\partial_i X^l + X^k \Gamma^l_{ik}] + g_{il}[\partial_j X^l + X^k \Gamma^l_{jk}] .\end{aligned}$$

Thus

$$(\mathcal{L} X)_{ij} = g_{jl}[\partial_i X^l + X^k \Gamma^l_{ik}] + g_{il}[\partial_j X^l + X^k \Gamma^l_{jk}] - \frac{2}{n}(\mathrm{div} X) g_{ij} . \tag{12.1}$$

Theorem 12.1. *Let (M^n, g) be a compact Riemannian manifold. Assume $g \in W^{s,p}(T^2 M)$ with $sp > n$ and $s \geq 1$. Suppose $e \in \mathbb{R}$ and $q \in (1, \infty)$ are such that for balls $\Omega \subseteq \mathbb{R}^n$ or for $\Omega = \mathbb{R}^n$*

$$\begin{aligned}&W^{s-1,p}(\Omega) \times W^{e+1,q}(\Omega) \hookrightarrow W^{e,q}(\Omega) , \\ &W^{s,p}(\Omega) \times W^{e,q}(\Omega) \hookrightarrow W^{e,q}(\Omega) , \\ &W^{s,p}(\Omega) \times W^{e+1,q}(\Omega) \hookrightarrow W^{e+1,q}(\Omega) .\end{aligned}$$

In the case that the above multiplication properties hold only for balls $\Omega \subseteq \mathbb{R}^n$ and not $\mathbb{R}^n$ itself, further assume that e and q are such that $\frac{\partial}{\partial x^j} : W^{e+1,q}(\Omega) \to W^{e,q}(\Omega)$

($1\leq j\leq n$) is continuous. Suppose $\{g_m\}$ is a sequence of smooth (C^∞) metrics on M such that $g_m\to g$ in $W^{s,p}(T^2M)$. Then

$$\mathcal{L}_{g_m}\to\mathcal{L}_g \quad \textit{in } L(W^{e+1,q}(TM),W^{e,q}(T^2M))\,.$$

Proof. In this proof we do not use the summation convention. Let $\Lambda=\{(U_\alpha,\varphi_\alpha,\rho_\alpha)\}_{\alpha=1}^N$ and $\tilde{\Lambda}=\{(U_\alpha,\varphi_\alpha,\tilde{\rho}_\alpha)\}_{\alpha=1}^N$ be standard total trivialization atlases for TM and T^2M, respectively. Without loss of generality we may assume that each of Λ and $\tilde{\Lambda}$ is super nice (or nice) and GL compatible with itself. Using Equation 12.1 and techniques discussed in Appendix A, one can show that under the hypotheses of the theorem, $\mathcal{L}_{g_m}$ and $\mathcal{L}_g$ indeed belong to $L(W^{e+1,q}(TM),W^{e,q}(T^2M))$.

Let $\{\psi_\alpha\}_{\alpha=1}^N$ be a partition of unity subordinate to the open cover $\{U_\alpha\}_{\alpha=1}^N$. Let $\tilde{\psi}_\alpha=\frac{\psi_\alpha^2}{\sum_{\beta=1}^N\psi_\beta^2}$. Note that $\frac{1}{\sum_{\beta=1}^N\psi_\beta^2}\circ\varphi_\alpha^{-1}\in BC^\infty(\varphi_\alpha(U_\alpha))$. We have

$$\|\mathcal{L}_{g_m}-\mathcal{L}_g\|_{op}=\sup_{\|X\|_{e+1,q}\neq 0,X\in C^\infty}\frac{\|(\mathcal{L}_{g_m}-\mathcal{L}_g)X\|_{e,q}}{\|X\|_{e+1,q}}\,.$$

Note that

$$\|(\mathcal{L}_{g_m}-\mathcal{L}_g)X\|_{W^{e,q}(T^2M)}\simeq\sum_{\alpha=1}^N\sum_{i,j=1}^n\|\tilde{\psi}_\alpha((\mathcal{L}_{g_m}-\mathcal{L}_g)X)_{ij}\circ\varphi_\alpha^{-1}\|_{W^{e,q}(\varphi_\alpha(U_\alpha))}\,.$$

By equation 12.1 we have

$$(\mathcal{L}_{g_m}X)_{ij}-(\mathcal{L}_gX)_{ij}=\sum_{l=1}^n[(g_m)_{jl}-g_{jl}]\partial_iX^l+\sum_{k,l=1}^n[(g_m)_{jl}(\Gamma_{g_m})_{ik}^l-g_{jl}(\Gamma_g)_{ik}^l]X^k$$
$$+\sum_{l=1}^n[(g_m)_{il}-g_{il}]\partial_jX^l+\sum_{k,l=1}^n[(g_m)_{il}(\Gamma_{g_m})_{jk}^l-g_{il}(\Gamma_g)_{jk}^l]X^k-\frac{2}{n}[(\mathrm{div}_{g_m}X)(g_m)_{ij}-(\mathrm{div}_gX)g_{ij}]\,.$$

Therefore,

$$\begin{aligned}\|(\mathcal{L}_{g_m}-\mathcal{L}_g)X\|_{W^{e,q}(T^2M)}\preceq&\\ \sum_{\alpha=1}^N\sum_{i,j,k,l=1}^n&\|\tilde{\psi}_\alpha[(g_m)_{jl}-g_{jl}]\partial_iX^l\circ\varphi_\alpha^{-1}\|_{e,q}\\ &+\|\tilde{\psi}_\alpha[(g_m)_{jl}(\Gamma_{g_m})_{ik}^l-g_{jl}(\Gamma_g)_{ik}^l]X^k\circ\varphi_\alpha^{-1}\|_{e,q}\\ &+\|\tilde{\psi}_\alpha[(g_m)_{il}-g_{il}]\partial_jX^l\circ\varphi_\alpha^{-1}\|_{e,q}+\|\tilde{\psi}_\alpha[(g_m)_{il}(\Gamma_{g_m})_{jk}^l-g_{il}(\Gamma_g)_{jk}^l]X^k\circ\varphi_\alpha^{-1}\|_{e,q}\\ &+\frac{2}{n}\|\tilde{\psi}_\alpha[(\mathrm{div}_{g_m}X)(g_m)_{ij}-(\mathrm{div}_gX)g_{ij}]\circ\varphi_\alpha^{-1}\|_{e,q}\,.\end{aligned}$$

Now, we consider each summand separately:

(1)

$$\|\tilde{\psi}_\alpha[(g_m)_{jl}-g_{jl}]\partial_iX^l\circ\varphi_\alpha^{-1}\|_{e,q}\preceq\|\psi_\alpha[(g_m)_{jl}-g_{jl}]\circ\varphi_\alpha^{-1}\|_{s,p}\|\psi_\alpha\partial_iX^l\circ\varphi_\alpha^{-1}\|_{e,q}\,.$$

Note that,

$$\begin{aligned}\|\psi_\alpha\partial_iX^l\circ\varphi_\alpha^{-1}\|_{e,q}&=\|\psi_\alpha\frac{\partial}{\partial x^i}(X^l\circ\varphi_\alpha^{-1})\|_{e,q}\\ &\leq\|\frac{\partial}{\partial x^i}\left[(\psi_\alpha\circ\varphi_\alpha^{-1})(X^l\circ\varphi_\alpha^{-1})\right]\|_{e,q}+\|[\frac{\partial}{\partial x^i}(\psi_\alpha\circ\varphi_\alpha^{-1})](X^l\circ\varphi_\alpha^{-1})\|_{e,q}\\ &\preceq\|\psi_\alpha X^l\circ\varphi_\alpha^{-1}\|_{e+1,q}+\|X\|_{e,q}\quad\text{(see Theorem 3.21)}\\ &\preceq\|X\|_{e+1,q}\,.\end{aligned}$$

Also,

$$\|\psi_\alpha[(g_m)_{jl}-g_{jl}]\circ\varphi_\alpha^{-1}\|_{s,p}\preceq\|g_m-g\|_{s,p}\,.$$

(2)

$$\begin{aligned}&\|\tilde{\psi}_\alpha[(g_m)_{jl}(\Gamma_{g_m})^l_{ik}-g_{jl}(\Gamma_g)^l_{ik}]X^k\circ\varphi_\alpha^{-1}\|_{e,q}\\&\quad\preceq\|\psi_\alpha[(g_m)_{jl}(\Gamma_{g_m})^l_{ik}-g_{jl}(\Gamma_g)^l_{ik}]\circ\varphi_\alpha^{-1}\|_{s-1,p}\|\psi_\alpha X^k\circ\varphi_\alpha^{-1}\|_{e+1,q}\\&\quad\preceq\|\psi_\alpha[(g_m)_{jl}(\Gamma_{g_m})^l_{ik}-g_{jl}(\Gamma_g)^l_{ik}]\circ\varphi_\alpha^{-1}\|_{s-1,p}\|X\|_{e+1,q}\,.\end{aligned}$$

(3)

$$\begin{aligned}&\|\tilde{\psi}_\alpha[(g_m)_{il}-g_{il}]\partial_jX^l\circ\varphi_\alpha^{-1}\|_{e,q}\preceq\|\psi_\alpha[(g_m)_{il}-g_{il}]\circ\varphi_\alpha^{-1}\|_{s,p}\|\psi_\alpha\partial_jX^l\circ\varphi_\alpha^{-1}\|_{e,q}\\&\preceq\|g_m-g\|_{s,p}\|X\|_{e+1,q}\quad\text{(see the procedure in item (1))}\,.\end{aligned}$$

(4)

$$\begin{aligned}&\|\tilde{\psi}_\alpha[(g_m)_{il}(\Gamma_{g_m})^l_{jk}-g_{il}(\Gamma_g)^l_{jk}]X^k\circ\varphi_\alpha^{-1}\|_{e,q}\\&\quad\preceq\|\psi_\alpha[(g_m)_{il}(\Gamma_{g_m})^l_{jk}-g_{il}(\Gamma_g)^l_{jk}]\circ\varphi_\alpha^{-1}\|_{s-1,p}\|\psi_\alpha X^k\circ\varphi_\alpha^{-1}\|_{e+1,q}\\&\quad\preceq\|\psi_\alpha[(g_m)_{il}(\Gamma_{g_m})^l_{jk}-g_{il}(\Gamma_g)^l_{jk}]\circ\varphi_\alpha^{-1}\|_{s-1,p}\|X\|_{e+1,q}\,.\end{aligned}$$

(5)

$$\begin{aligned}&\|\tilde{\psi}_\alpha[(\text{div}_{g_m}X)(g_m)_{ij}-(\text{div}_gX)g_{ij}]\circ\varphi_\alpha^{-1}\|_{e,q}=\\&\|\tilde{\psi}_\alpha[(\text{div}_{g_m}X)(g_m)_{ij}-(\text{div}_gX)(g_m)_{ij}+(\text{div}_gX)(g_m)_{ij}-(\text{div}_gX)g_{ij}]\circ\varphi_\alpha^{-1}\|_{e,q}\\&\preceq\|(\text{div}_{g_m}X)-(\text{div}_gX)\|_{e,q}\|\tilde{\psi}_\alpha(g_m)_{ij}\circ\varphi_\alpha^{-1}\|_{s,p}+\|\text{div}_gX\|_{e,q}\|\tilde{\psi}_\alpha((g_m)_{ij}-g_{ij})\circ\varphi_\alpha^{-1}\|_{s,p}\\&\preceq\|(\text{div}_{g_m})-(\text{div}_g)\|_{op}\|X\|_{e+1,q}\|g_m\|_{s,p}+\|\text{div}_g\|_{op}\|X\|_{e+1,q}\|g_m-g\|_{s,p}\,.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\|\mathcal{L}_{g_m}-\mathcal{L}_g\|_{op}&=\sup_{\|X\|_{e+1,q}\neq0,X\in C^\infty}\frac{\|(\mathcal{L}_{g_m}-\mathcal{L}_g)X\|_{e,q}}{\|X\|_{e+1,q}}\preceq\sum_{\alpha=1}^N\sum_{i,j,k,l=1}^n(2+\|\text{div}_g\|_{op})\|g_m-g\|_{s,p}\\&+\|\psi_\alpha[(g_m)_{jl}(\Gamma_{g_m})^l_{ik}-g_{jl}(\Gamma_g)^l_{ik}]\circ\varphi_\alpha^{-1}\|_{s-1,p}+\|\psi_\alpha[(g_m)_{il}(\Gamma_{g_m})^l_{jk}-g_{il}(\Gamma_g)^l_{jk}]\circ\varphi_\alpha^{-1}\|_{s-1,p}\\&+\|(\text{div}_{g_m})-(\text{div}_g)\|_{op}\|g_m\|_{s,p}\,.\end{aligned}$$

Now, note that

- Under the hypotheses of this theorem, $\text{div}_g:W^{e+1,q}(TM)\to W^{e,q}(M)$ is a continuous linear operator (see Example 3 in Appendix A). Therefore, $\|\text{div}_g\|_{op}$ is a finite number.
- By assumption $\|g_m-g\|_{s,p}\to0$.
- As a consequence of Theorem 4.3 we have

$$\begin{aligned}&(g_m)_{jl}\circ\varphi_\alpha^{-1}\to g_{jl}\circ\varphi_\alpha^{-1}\quad\text{in }W^{s,p}_{loc}(\varphi_\alpha(U_\alpha))\,,\\&(\Gamma_{g_m})^l_{ik}\circ\varphi_\alpha^{-1}\to(\Gamma_g)^l_{ik}\circ\varphi_\alpha^{-1}\quad\text{in }W^{s-1,p}_{loc}(\varphi_\alpha(U_\alpha))\,.\end{aligned}$$

Since $W^{s,p}_{loc}\times W^{s-1,p}_{loc}\hookrightarrow W^{s-1,p}_{loc}$, we get

$$(g_m)_{jl}(\Gamma_{g_m})^l_{ik}\circ\varphi_\alpha^{-1}\to g_{jl}(\Gamma_g)^l_{ik}\circ\varphi_\alpha^{-1}\quad\text{in }W^{s-1,p}_{loc}(\varphi_\alpha(U_\alpha))\,,$$

which implies that

$$\|\psi_\alpha[(g_m)_{jl}(\Gamma_{g_m})^l_{ik}-g_{jl}(\Gamma_g)^l_{ik}]\circ\varphi_\alpha^{-1}\|_{s-1,p}\to0\,.$$

Similarly,

$$\|\psi_\alpha[(g_m)_{il}(\Gamma_{g_m})^l_{jk}-g_{il}(\Gamma_g)^l_{jk}]\circ\varphi_\alpha^{-1}\|_{s-1,p}\to0\,.$$

- It follows from Example 3 in Appendix A and Theorem 10.1 that
$$\text{div}_{g_m} \to \text{div}_g \quad \text{in } L(W^{e+1,q}(TM), W^{e,q}(M))\,.$$
Also, since $g_m \to g$ in $W^{s,p}(T^2M)$, $\|g_m\|_{s,p}$ is bounded.

Thus $\|\mathcal{L}_{g_m} - \mathcal{L}_g\|_{op} \to 0$ as $m \to \infty$. □

13. Vector Laplacian with Rough Metric

div$\mathcal{L}$ is sometimes called *vector Laplacian* and is denoted by Δ_L.

Theorem 13.1. *Let (M^n, g) be a compact Riemannian manifold. Assume $g \in W^{s,p}(T^2M)$ with $sp > n$ and $s \geq 1$. Suppose $e \in \mathbb{R}$ and $q \in (1,\infty)$ are such that for balls $\Omega \subseteq \mathbb{R}^n$ or for $\Omega = \mathbb{R}^n$*
$$\begin{aligned}
&W^{s-1,p}(\Omega) \times W^{e+1,q}(\Omega) \hookrightarrow W^{e,q}(\Omega)\,,\\
&W^{s,p}(\Omega) \times W^{e,q}(\Omega) \hookrightarrow W^{e,q}(\Omega)\,,\\
&W^{s,p}(\Omega) \times W^{e+1,q}(\Omega) \hookrightarrow W^{e+1,q}(\Omega)\,,\\
&W^{s-1,p}(\Omega) \times W^{e,q}(\Omega) \hookrightarrow W^{e-1,q}(\Omega)\,,\\
&W^{s,p}(\Omega) \times W^{e-1,q}(\Omega) \hookrightarrow W^{e-1,q}(\Omega)\,.
\end{aligned}$$
In the case that the above multiplication properties hold only for balls $\Omega \subseteq \mathbb{R}^n$ and not $\mathbb{R}^n$ itself, further assume that e and q are such that $\frac{\partial}{\partial x^j} : W^{e+1,q}(\Omega) \to W^{e,q}(\Omega)$ and $\frac{\partial}{\partial x^j} : W^{e,q}(\Omega) \to W^{e-1,q}(\Omega)$ $(1 \leq j \leq n)$ are continuous.

Suppose $\{g_m\}$ is a sequence of smooth (C^∞) metrics on M such that $g_m \to g$ in $W^{s,p}(T^2M)$. Then
$$(\Delta_L)_{g_m} \to (\Delta_L)_g \quad \textit{in } L(W^{e+1,q}(TM), W^{e-1,q}(T^1M))\,.$$

Proof. By Theorem 12.1,
$$\mathcal{L}_{g_m} \to \mathcal{L}_g \qquad \text{in } L(W^{e+1,q}(TM) \to W^{e,q}(T^2M))\,.$$
Also, by Theorem 10.3,
$$\text{div}_{g_m} \to \text{div}_g \qquad \text{in } L(W^{e,q}(T^2M) \to W^{e-1,q}(T^1M))\,.$$
Therefore, it follows from Theorem 3.4 that
$$\text{div}_{g_m} \circ \mathcal{L}_{g_m} \to \text{div}_g \circ \mathcal{L}_g \qquad \text{in } L(W^{e+1,q}(TM) \to W^{e-1,q}(T^1M))\,.$$
□

14. Curvature with Rough Metric

Let (M^n, g) be a Riemannian manifold. The *Riemannian curvature tensor* is the covariant 4-tensor field defined by
$$\text{Rm}(X,Y,Z,W) = g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W)\,.$$
With respect to any local chart (U,φ) we have $[\partial_i, \partial_j] = 0$ and
$$\begin{aligned}
\nabla_i\nabla_j\partial_k &= \nabla_i(\Gamma^r_{jk}\partial_r) = \partial_i(\Gamma^r_{jk})\partial_r + \Gamma^r_{jk}\Gamma^l_{ir}\partial_l\\
&= \left[\partial_i\Gamma^p_{jk} + \Gamma^r_{jk}\Gamma^p_{ir}\right]\partial_p\,.
\end{aligned}$$
Therefore, by subtracting the same expression with i and j interchanged we get
$$\nabla_i\nabla_j\partial_k - \nabla_j\nabla_i\partial_k = \left[\partial_i\Gamma^p_{jk} - \partial_j\Gamma^p_{ik} + \Gamma^r_{jk}\Gamma^p_{ir} - \Gamma^r_{ik}\Gamma^p_{jr}\right]\partial_p\,.$$

Subsequently,

$$\begin{aligned} R_{ijkl} &= \text{Rm}(\partial_i, \partial_j, \partial_k, \partial_l) = g(\nabla_i \nabla_j \partial_k - \nabla_j \nabla_i \partial_k, \partial_l) \\ &= g_{pl}\left[\partial_i \Gamma^p_{jk} - \partial_j \Gamma^p_{ik} + \Gamma^r_{jk}\Gamma^p_{ir} - \Gamma^r_{ik}\Gamma^p_{jr}\right]. \end{aligned}$$

The *Ricci tensor* is the covariant 2-tensor field defined by

$$\text{Ric} = \text{tr}(\text{sharp}_g \text{Rm}).$$

where the trace is on the leftmost covariant component and the only contravariant component of sharp_gRm. With respect to any local coordinate chart

$$\text{Ric}_{ij} := g^{km} R_{kijm}.$$

The *scalar curvature* Scal is the function defined as the trace of the Ricci tensor

$$\text{Scal} := \text{tr}\left(\text{sharp}_g \text{Ric}\right).$$

Theorem 14.1. *Let (M^n, g) be a compact Riemannian manifold. Assume $g \in W^{s,p}(T^2M)$ with $sp > n$, $s \geq 2$, and $n \geq 2$. Then Rm belongs to $W^{s-2,p}(T^4M)$, Ric belongs to $W^{s-2,p}(T^2M)$, and Scal belongs to $W^{s-2,p}(M)$.*

Proof. Let $\{(U_\alpha, \varphi_\alpha)\}_{1\leq\alpha\leq N}$ be an atlas for M. By Theorem 3.20 it is enough to show that for each $1 \leq \alpha \leq N$ and $1 \leq i, j, k, l \leq n$

$$\text{Rm}_{ijkl} \circ \varphi_\alpha^{-1} \in W^{s-2,p}_{loc}(\varphi_\alpha(U_\alpha)).$$

Recall that,

$$\text{Rm}_{ijkl} \circ \varphi_\alpha^{-1} = g_{pl}\left[\partial_i \Gamma^p_{jk} - \partial_j \Gamma^p_{ik} + \Gamma^r_{jk}\Gamma^p_{ir} - \Gamma^r_{ik}\Gamma^p_{jr}\right] \circ \varphi_\alpha^{-1}.$$

By Corollary 3.23, Theorem 3.25, and Theorem 3.11 we have

$$g_{pl} \circ \varphi_\alpha^{-1} \in W^{s,p}_{loc}(\varphi_\alpha(U_\alpha)), \quad \partial_i \Gamma^p_{jk} \circ \varphi_\alpha^{-1}, \partial_j \Gamma^p_{ik} \circ \varphi_\alpha^{-1} \in W^{s-2,p}_{loc}(\varphi_\alpha(U_\alpha)).$$

Also, considering Theorem 3.14, since $W^{s-1,p} \times W^{s-1,p} \hookrightarrow W^{s-2,p}$, we have

$$\Gamma^r_{jk}\Gamma^p_{ir} \circ \varphi_\alpha^{-1} \in W^{s-2,p}_{loc}(\varphi_\alpha(U_\alpha)), \quad \Gamma^r_{ik}\Gamma^p_{jr} \circ \varphi_\alpha^{-1} \in W^{s-2,p}_{loc}(\varphi_\alpha(U_\alpha)).$$

Finally, since $W^{s,p} \times W^{s-2,p} \hookrightarrow W^{s-2,p}$,

$$g_{pl}\left[\partial_i \Gamma^p_{jk} - \partial_j \Gamma^p_{ik} + \Gamma^r_{jk}\Gamma^p_{ir} - \Gamma^r_{ik}\Gamma^p_{jr}\right] \circ \varphi_\alpha^{-1} \in W^{s-2,p}_{loc}(\varphi_\alpha(U_\alpha)).$$

So, $\text{Rm} \in W^{s-2,p}(T^4M)$.
Since $W^{s,p} \times W^{s-2,p} \hookrightarrow W^{s-2,p}$, it follows from Theorem 5.3 that $\text{sharp}_g : W^{s-2,p}(T^4M) \to W^{s-2,p}(T^3_1M)$ is well-defined and continuous. Also, by Theorem 9.1, $\text{tr} : W^{s-2,p}(T^3_1M) \to W^{s-2,p}(T^2M)$ is well-defined and continuous. Therefore, $\text{Ric} = \text{tr}(\text{sharp}_g\text{Rm})$ belongs to $W^{s-2,p}(T^2M)$.
The same argument shows that $\text{Scal} := \text{tr}\left(\text{sharp}_g \text{Ric}\right)$ must belong to $W^{s-2,p}(M)$. □

Theorem 14.2. *Let (M^n, g) be a compact Riemannian manifold. Assume $g \in W^{s,p}(T^2M)$ with $sp > n$, $s \geq 2$, and $n \geq 2$. . Suppose $\{g_m\}$ is a sequence of smooth (C^∞) metrics on M such that $g_m \to g$ in $W^{s,p}(T^2M)$. Then*

$$Rm_{g_m} \to Rm_g \quad \text{in } W^{s-2,p}(T^4M).$$

Proof. In this proof we will not use the summation convention. Let $\{(U_\alpha,\varphi_\alpha)\}_{1\leq\alpha\leq N}$ be a super nice atlas for M that is GL compatible with itself and $\{\psi_\alpha\}$ be a subordinate partition of unity. We have

$$\|\mathrm{Rm}_{g_m}-\mathrm{Rm}_g\|_{s-2,p}\simeq\sum_{\alpha=1}^{N}\sum_{i,j,k,l=1}^{n}\|\psi_\alpha(\mathrm{Rm}_{g_m}-\mathrm{Rm}_g)_{ijkl}\circ\varphi_\alpha^{-1}\|_{W^{s-2,p}(\varphi_\alpha(U_\alpha))}$$

$$\begin{aligned}&\preceq\sum_{\alpha=1}^{N}\sum_{i,j,k,l,p,r=1}^{n}\|\psi_\alpha((g_m)_{pl}\partial_i(\Gamma_{g_m})^p_{jk}-g_{pl}\partial_i(\Gamma_g)^p_{jk})\circ\varphi_\alpha^{-1}\|_{W^{s-2,p}(\varphi_\alpha(U_\alpha))}\\&+\|\psi_\alpha((g_m)_{pl}\partial_j(\Gamma_{g_m})^p_{ik}-g_{pl}\partial_j(\Gamma_g)^p_{ik})\circ\varphi_\alpha^{-1}\|_{W^{s-2,p}(\varphi_\alpha(U_\alpha))}\\&+\|\psi_\alpha((\Gamma_{g_m})^r_{jk}(\Gamma_{g_m})^p_{ir}-\Gamma^r_{jk}\Gamma^p_{ir})\circ\varphi_\alpha^{-1}\|_{W^{s-2,p}(\varphi_\alpha(U_\alpha))}\\&+\|\psi_\alpha((\Gamma_{g_m})^r_{ik}(\Gamma_{g_m})^p_{jr}-\Gamma^r_{ik}\Gamma^p_{jr})\circ\varphi_\alpha^{-1}\|_{W^{s-2,p}(\varphi_\alpha(U_\alpha))}\,.\end{aligned}$$

We consider each term separately:

(1) By Theorem 4.3 $(\Gamma_{g_m})^p_{jk}\circ\varphi_\alpha^{-1}\to(\Gamma_g)^p_{jk}\circ\varphi_\alpha^{-1}$ in $W^{s-1,p}_{loc}$ and $(g_m)_{pl}\circ\varphi_\alpha^{-1}\to g_{pl}\circ\varphi_\alpha^{-1}$ in $W^{s,p}_{loc}$. It follows from Theorem 3.11 that $\partial_i(\Gamma_{g_m})^p_{jk}\circ\varphi_\alpha^{-1}\to\partial_i(\Gamma_g)^p_{jk}\circ\varphi_\alpha^{-1}$ in $W^{s-2,p}_{loc}$ and subsequently since $W^{s,p}\times W^{s-2,p}\hookrightarrow W^{s-2,p}$ we get

$$(g_m)_{pl}\partial_i(\Gamma_{g_m})^p_{jk}\circ\varphi_\alpha^{-1}\to g_{pl}\partial_i(\Gamma_g)^p_{jk}\circ\varphi_\alpha^{-1}\quad\text{in } W^{s-2,p}_{loc}(\varphi_\alpha(U_\alpha))\,.$$

Therefore,

$$\|\psi_\alpha((g_m)_{pl}\partial_i(\Gamma_{g_m})^p_{jk}-g_{pl}\partial_i(\Gamma_g)^p_{jk})\circ\varphi_\alpha^{-1}\|_{W^{s-2,p}(\varphi_\alpha(U_\alpha))}\to 0\quad\text{as } m\to\infty\,.$$

(2) Interchanging the roles of i and j in the above argument shows that

$$\|\psi_\alpha((g_m)_{pl}\partial_j(\Gamma_{g_m})^p_{ik}-g_{pl}\partial_j(\Gamma_g)^p_{ik})\circ\varphi_\alpha^{-1}\|_{W^{s-2,p}(\varphi_\alpha(U_\alpha))}\to 0\quad\text{as } m\to\infty\,.$$

(3) By Theorem 4.3,

$$(\Gamma_{g_m})^r_{jk}\circ\varphi_\alpha^{-1}\to(\Gamma_g)^r_{jk}\circ\varphi_\alpha^{-1},\qquad(\Gamma_{g_m})^p_{ir}\circ\varphi_\alpha^{-1}\to(\Gamma_g)^p_{ir}\circ\varphi_\alpha^{-1}\quad\text{in } W^{s-1,p}_{loc}\,.$$

Since $W^{s-1,p}\times W^{s-1,p}\hookrightarrow W^{s-2,p}$, we obtain

$$(\Gamma_{g_m})^r_{jk}(\Gamma_{g_m})^p_{ir}\circ\varphi_\alpha^{-1}\to(\Gamma_g)^r_{jk}(\Gamma_g)^p_{ir}\circ\varphi_\alpha^{-1}\quad\text{in } W^{s-2,p}_{loc}\,.$$

Therefore,

$$\|\psi_\alpha((\Gamma_{g_m})^r_{jk}(\Gamma_{g_m})^p_{ir}-\Gamma^r_{jk}\Gamma^p_{ir})\circ\varphi_\alpha^{-1}\|_{W^{s-2,p}(\varphi_\alpha(U_\alpha))}\to 0\quad\text{as } m\to\infty\,.$$

(4) Interchanging the roles of i and j in the above argument shows that

$$\|\psi_\alpha((\Gamma_{g_m})^r_{ik}(\Gamma_{g_m})^p_{jr}-\Gamma^r_{ik}\Gamma^p_{jr})\circ\varphi_\alpha^{-1}\|_{W^{s-2,p}(\varphi_\alpha(U_\alpha))}\to 0\quad\text{as } m\to\infty\,.$$

Hence

$$\|\mathrm{Rm}_{g_m}-\mathrm{Rm}_g\|_{s-2,p}\to 0\,.$$

□

Theorem 14.3. *Let (M^n,g) be a compact Riemannian manifold. Assume $g\in W^{s,p}(T^2M)$ with $sp>n$, $s\geq 2$, and $n\geq 2$. Suppose $\{g_m\}$ is a sequence of smooth (C^∞) metrics on M such that $g_m\to g$ in $W^{s,p}(T^2M)$. Then*

$$Ric_{g_m}\to Ric_g\quad\text{in } W^{s-2,p}(T^2M)\,.$$

Proof. By Theorem 14.2, $\mathrm{Rm}_{g_m}\to\mathrm{Rm}_g$ in $W^{s-2,p}(T^4M)$. Also it follows from the hypotheses of the theorem that $W^{s,p}\times W^{s-2,p}\to W^{s-2,p}$. Thus by Theorem 5.4,

$$\mathrm{sharp}_{g_m}\to\mathrm{sharp}_g\quad\text{in } L(W^{s-2,p}(T^4M),W^{s-2,p}(T^3_1M))\,.$$

Consequently,

$$\mathrm{sharp}_{g_m}(\mathrm{Rm}_{g_m}) \to \mathrm{sharp}_g(\mathrm{Rm}_g) \quad \text{in } W^{s-2,p}(T^3_1 M)\,.$$

Now, it follows from Theorem 9.1 that

$$\mathrm{tr}\,\mathrm{sharp}_{g_m}(\mathrm{Rm}_{g_m}) \to \mathrm{tr}\,\mathrm{sharp}_g(\mathrm{Rm}_g) \quad \text{in } W^{s-2,p}(T^2 M)\,.$$

That is,

$$\mathrm{Ric}_{g_m} \to \mathrm{Ric}_g \quad \text{in } W^{s-2,p}(T^2 M)\,.$$

□

Theorem 14.4. *Let (M^n, g) be a compact Riemannian manifold. Assume $g \in W^{s,p}(T^2 M)$ with $sp > n$, $s \geq 2$, and $n \geq 2$. Suppose $\{g_m\}$ is a sequence of smooth (C^∞) metrics on M such that $g_m \to g$ in $W^{s,p}(T^2 M)$. Then*

$$\mathit{Scal}_{g_m} \to \mathit{Scal}_g \quad \text{in } W^{s-2,p}(M)\,.$$

Proof. By Theorem 14.3, $\mathrm{Ric}_{g_m} \to \mathrm{Ric}_g$ in $W^{s-2,p}(T^2 M)$. Also it follows from the hypotheses of the theorem that $W^{s,p} \times W^{s-2,p} \to W^{s-2,p}$. Thus by Theorem 5.4,

$$\mathrm{sharp}_{g_m} \to \mathrm{sharp}_g \quad \text{in } L(W^{s-2,p}(T^2 M), W^{s-2,p}(T^1_1 M))\,.$$

Consequently,

$$\mathrm{sharp}_{g_m}(\mathrm{Ric}_{g_m}) \to \mathrm{sharp}_g(\mathrm{Ric}_g) \quad \text{in } W^{s-2,p}(T^1_1 M)\,.$$

Now, it follows from Theorem 9.1 that

$$\mathrm{tr}\,\mathrm{sharp}_{g_m}(\mathrm{Rm}_{g_m}) \to \mathrm{tr}\,\mathrm{sharp}_g(\mathrm{Rm}_g) \quad \text{in } W^{s-2,p}(M)\,.$$

That is,

$$\mathrm{Scal}_{g_m} \to \mathrm{Scal}_g \quad \text{in } W^{s-2,p}(M)\,.$$

□

Appendix A. Differential Operators on Compact Manifolds

First we recite several definitions and facts from [7]. Let M^n be a compact smooth manifold. Let E and $\tilde{E}$ be two vector bundles over M of ranks r and $\tilde{r}$, respectively. A linear operator $P : C^\infty(M, E) \to \Gamma(M, \tilde{E})$ is called **local** if

$$\forall\, u \in C^\infty(M, E) \qquad \mathrm{supp}\, Pu \subseteq \mathrm{supp}\, u\,.$$

As it is discussed in [7], if P is a local operator, then it is possible to have a well-defined notion of restriction of P to open sets $U \subseteq M$, that is, if $P : C^\infty(M, E) \to \Gamma(M, \tilde{E})$ is local and $U \subseteq M$ is open, then we can define a map

$$P|_U : C^\infty(U, E_U) \to \Gamma(U, \tilde{E}_U)$$

with the property that

$$\forall\, u \in C^\infty(M, E) \qquad (Pu)|_U = P|_U(u|_U)\,.$$

For any nonempty set V in $\mathbb{R}^r$, let $\mathrm{Func}(V, \mathbb{R}^t)$ denote the vector space of all functions from V to $\mathbb{R}^t$. By the *local representation of* P with respect to the total trivialization triples (U, φ, ρ) of E and $(U, \varphi, \tilde{\rho})$ of $\tilde{E}$ we mean the linear transformation $Q : C^\infty(\varphi(U), \mathbb{R}^r) \to \mathrm{Func}(\varphi(U), \mathbb{R}^{\tilde{r}})$ defined by

$$Q(f) = \tilde{\rho} \circ P(\rho^{-1} \circ f \circ \varphi) \circ \varphi^{-1}\,.$$

If we denote the components of $f \in C^\infty(\varphi(U), \mathbb{R}^r)$ by $(f^1, \cdots, f^r)$, then we can write $Q(f^1, \cdots, f^r) = (h^1, \cdots, h^{\tilde{r}})$ where for all $1 \leq k \leq \tilde{r}$

$$h^k = \pi_k \circ Q(f^1, \cdots, f^r) \overset{Q \text{ is linear}}{=} \pi_k \circ Q(f^1, 0, \cdots, 0) + \cdots + \pi_k \circ Q(0, \cdots, 0, f^r)\,.$$

So, if for each $1 \leq k \leq \tilde{r}$ and $1 \leq i \leq r$ we define $Q_{ki} : C^\infty(\varphi(U), \mathbb{R}) \to \text{Func}(\varphi(U), \mathbb{R})$ by

$$Q_{ki}(g) = \pi_k \circ Q(0, \cdots, 0, \underbrace{g}_{i^{th} \text{ position}}, 0, \cdots, 0)\,,$$

then we have

$$Q(f^1, \cdots, f^r) = (\sum_{i=1}^{r} Q_{1i}(f^i), \cdots, \sum_{i=1}^{r} Q_{\tilde{r}i}(f^i))\,.$$

Results of the following type are discussed in [7].

Theorem A.1. *([7], Page 100) Let M^n be a compact smooth manifold. Let $P : C^\infty(M, E) \to \Gamma(M, \tilde{E})$ be a local operator. Let $\Lambda = \{(U_\alpha, \varphi_\alpha, \rho_\alpha, \psi_\alpha)\}_{1 \leq \alpha \leq N}$ and $\tilde{\Lambda} = \{(U_\alpha, \varphi_\alpha, \tilde{\rho}_\alpha, \psi_\alpha)\}_{1 \leq \alpha \leq N}$ be two augmented total trivialization atlases for E and $\tilde{E}$, respectively. Suppose the atlas $\{(U_\alpha, \varphi_\alpha)\}_{1 \leq \alpha \leq N}$ is GL compatible with itself. For each $1 \leq \alpha \leq N$, let Q^α denote the local representation of P with respect to the total trivialization triples $(U_\alpha, \varphi_\alpha, \rho_\alpha)$ and $(U_\alpha, \varphi_\alpha, \tilde{\rho}_\alpha)$ of E and $\tilde{E}$, respectively. Suppose $e, \tilde{e} \in \mathbb{R}$, $1 < q, \tilde{q} < \infty$, and for each $1 \leq \alpha \leq N$, $1 \leq i \leq \tilde{r}$, and $1 \leq j \leq r$,*

$$Q^\alpha_{ij} : (C^\infty_c(\varphi_\alpha(U_\alpha)), \|.\|_{e,q}) \to W^{\tilde{e},\tilde{q}}_{loc}(\varphi_\alpha(U_\alpha))$$

is well-defined and continuous and does not increase support. Then

- *$P(C^\infty(M, E)) \subseteq W^{\tilde{e},\tilde{q}}(M, \tilde{E}; \tilde{\Lambda})$,*
- *$P : (C^\infty(M, E), \|.\|_{e,q}) \to W^{\tilde{e},\tilde{q}}(M, \tilde{E}; \tilde{\Lambda})$ is continuous and so it can be extended to a continuous linear map $P : W^{e,q}(M, E; \Lambda) \to W^{\tilde{e},\tilde{q}}(M, \tilde{E}; \tilde{\Lambda})$.*

In the following examples we assume (M^n, g) is a compact Riemannian manifold with $g \in W^{s,p}(M, T^2M)$, $sp > n$, and $s \geq 1$. The local representations are all assumed to be with respect to charts in a super nice total trivialization atlas that is GL compatible with itself. The first example is taken from [7].

- **Example 1: Differential** Consider $d : C^\infty(M) \to C^\infty(T^*M)$. The local representation of d is $Q : C^\infty(\varphi(U)) \to C^\infty(\varphi(U), \mathbb{R}^n)$ which is defined by

$$\begin{aligned} Q(f)(a) &= \tilde{\rho} \circ d(\rho^{-1} \circ f \circ \varphi) \circ \varphi^{-1}(a) \\ &= \tilde{\rho} \circ (\frac{\partial f}{\partial x^i}|_{\varphi(\varphi^{-1}(a))} dx^i|_{\varphi^{-1}(a)}) \\ &= (\frac{\partial f}{\partial x^1}|_a, \cdots, \frac{\partial f}{\partial x^n}|_a)\,. \end{aligned}$$

Here we used $\rho = Id$ and the fact that if $g : M \to \mathbb{R}$ is smooth, then

$$(dg)(p) = \frac{\partial(g \circ \varphi^{-1})}{\partial x^i}|_{\varphi(p)} dx^i|_p\,.$$

Clearly, each component of Q is a continuous operator from $(C^\infty_c(\varphi(U)), \|.\|_{e,q})$ to $W^{e-1,q}(\varphi(U)) \hookrightarrow W^{e-1,q}_{loc}(\varphi(U))$ (see Theorem 3.9; note that $\varphi(U) = \mathbb{R}^n$). Hence d can be viewed as a continuous operator from $W^{e,q}(M)$ to $W^{e-1,q}(T^*M)$.

- **Example 2: Gradient** Suppose e and q are such that for balls $\Omega \subseteq \mathbb{R}^n$ or for $\Omega = \mathbb{R}^n$

$$W^{s,p}(\Omega) \times W^{e,q}(\Omega) \hookrightarrow W^{e,q}(\Omega)\,.$$

In section 5 we proved that $\text{sharp}_g : W^{e,q}(T^*M) \to W^{e,q}(TM)$ is well-defined and continuous. Also in the previous example we showed that for all e and q, $d : W^{e+1,q}(M) \to W^{e,q}(T^*M)$ is well-defined and continuous. Consequently, $\text{grad}_g : W^{e+1,q}(M) \to W^{e,q}(TM)$ defined by

$$\text{grad}_g = \text{sharp}_g \circ d$$

is also continuous.

- **Example 3: Divergence** Consider $\text{div} : C^\infty(TM) \to \text{Func}(M,\mathbb{R})$. Here we will show that if e and q are such that

$$W^{s,p}(\mathbb{R}^n) \times W^{e,q}(\mathbb{R}^n) \hookrightarrow W^{e,q}(\mathbb{R}^n) \tag{A.1}$$
$$W^{s,p}(\mathbb{R}^n) \times W^{e-1,q}(\mathbb{R}^n) \hookrightarrow W^{e-1,q}(\mathbb{R}^n)\,, \tag{A.2}$$

then div can be considered as a continuous operator from $W^{e,q}(TM)$ to $W^{e-1,q}(M)$. The local representation of divergence with respect to the coordinate chart (U,φ) is $Q : C^\infty(\varphi(U),\mathbb{R}^n) \to \text{Func}(\varphi(U),\mathbb{R})$ defined by

$$\begin{aligned} Q(Y) &= \tilde{\rho} \circ \text{div}(\rho^{-1} \circ Y \circ \varphi) \circ \varphi^{-1} \quad (Y : \varphi(U) \to \mathbb{R}^n, \quad Y = (Y^1, \cdots, Y^n)) \\ &= \text{div}((Y^1 \circ \varphi)\partial_1 + \cdots + (Y^n \circ \varphi)\partial_n) \circ \varphi^{-1} \\ &= \sum_{j=1}^n \frac{1}{\sqrt{\det g} \circ \varphi^{-1}} \frac{\partial}{\partial x^j}\big[(\sqrt{\det g} \circ \varphi^{-1})(Y^j)\big]\,. \end{aligned}$$

Note that in the above, $\tilde{\rho} = Id$ and

$$\rho^{-1}(Y \circ \varphi) = \rho^{-1}(Y^1 \circ \varphi, \cdots, Y^n \circ \varphi) = (Y^1 \circ \varphi)\partial_1 + \cdots + (Y^n \circ \varphi)\partial_n\,.$$

Moreover, we used the fact that for any vector field X defined on U

$$(\text{div}X) \circ \varphi^{-1} = \sum_{j=1}^n \frac{1}{\sqrt{\det g} \circ \varphi^{-1}} \frac{\partial}{\partial x^j}\big[(\sqrt{\det g} \circ \varphi^{-1})(X^j \circ \varphi^{-1})\big]\,.$$

Also, note that $Q(Y) = \sum_{j=1}^n Q_{1j}(Y^j)$ where $Q_{1j} : C^\infty(\varphi(U),\mathbb{R}) \to \text{Func}(\varphi(U),\mathbb{R})$ and for all $f \in C^\infty(\varphi(U),\mathbb{R})$, $Q_{1j}(f)$ is the first (the only) component of

$$Q(0,\cdots,0,\underbrace{f}_{j^{th}\text{ position}},0,\cdots,0)\,.$$

That is,

$$\forall\, 1 \le j \le n \qquad Q_{1j}(f) = \frac{1}{\sqrt{\det g} \circ \varphi^{-1}} \frac{\partial}{\partial x^j}\big[(\sqrt{\det g} \circ \varphi^{-1})(f)\big]\,.$$

Now, suppose $f \in C_c^\infty(\varphi(U))$. So, clearly, $f \in W^{e,q}_{loc}(\varphi(U))$. It follows from the hypotheses on e and q that (see Theorem 3.14)

$$\begin{aligned} &W^{s,p}_{loc}(\varphi(U)) \times W^{e,q}_{loc}(\varphi(U)) \hookrightarrow W^{e,q}_{loc}(\varphi(U))\,, \\ &W^{s,p}_{loc}(\varphi(U)) \times W^{e-1,q}_{loc}(\varphi(U)) \hookrightarrow W^{e-1,q}_{loc}(\varphi(U))\,. \end{aligned}$$

Also, by Theorem 3.24 we know that $\sqrt{\det g}\circ\varphi^{-1}$ and $\frac{1}{\sqrt{\det g\circ\varphi^{-1}}}$ are in $W^{s,p}_{loc}(\varphi(U))$. Hence we have the following chain of continuous maps

$$W^{e,q}_{loc}\to W^{e,q}_{loc}\to W^{e-1,q}_{loc}\to W^{e-1,q}_{loc}$$
$$f\mapsto(\sqrt{\det g}\circ\varphi^{-1})f\mapsto\frac{\partial}{\partial x^j}((\sqrt{\det g}\circ\varphi^{-1})f)\mapsto\frac{1}{\sqrt{\det g}\circ\varphi^{-1}}\frac{\partial}{\partial x^j}((\sqrt{\det g}\circ\varphi^{-1})f)$$

which proves the continuity of $Q_{1j}:(C^\infty_c(\varphi(U)),\|.\|_{e,q})\to W^{e-1,q}_{loc}(\varphi(U))$.

Remark A.2. *Instead of A.1 and A.2, we may alternatively assume that for all balls $\Omega\subseteq\mathbb{R}^n$*

$$W^{s,p}(\Omega)\times W^{e,q}(\Omega)\hookrightarrow W^{e,q}(\Omega)\,,$$
$$W^{s,p}(\Omega)\times W^{e-1,q}(\Omega)\hookrightarrow W^{e-1,q}(\Omega)\,,$$

and work with nice charts instead of super nice charts. However, if we do so, then we need to additionally assume that e and q are such that $\frac{\partial}{\partial x^j}:W^{e,q}(\Omega)\to W^{e-1,q}(\Omega)$ ($1\le j\le n$) is continuous (see Theorem 3.9).

- **Example 4: Lie Derivative** Let $X\in W^{\tilde{s},\tilde{p}}(TM)$. Consider $L_X:C^\infty(T^kM)\to\Gamma(T^kM)$. Here we will show that if e and q are such that

$$W^{\tilde{s},\tilde{p}}(\mathbb{R}^n)\times W^{e-1,q}(\mathbb{R}^n)\hookrightarrow W^{e-1,q}(\mathbb{R}^n) \tag{A.3}$$
$$W^{\tilde{s}-1,\tilde{p}}(\mathbb{R}^n)\times W^{e,q}(\mathbb{R}^n)\hookrightarrow W^{e-1,q}(\mathbb{R}^n)\,, \tag{A.4}$$

then L_X can be considered as a continuous operator from $W^{e,q}(T^kM)$ to $W^{e-1,q}(T^kM)$. The local representation of L_X with respect to the coordinate chart (U,φ) is $Q:C^\infty(\varphi(U),\mathbb{R}^{(n^k)})\to\text{Func}(\varphi(U),\mathbb{R}^{(n^k)})$ defined by

$$Q(F)=\rho\circ L_X(\rho^{-1}\circ F\circ\varphi)\circ\varphi^{-1}\quad(F:\varphi(U)\to\mathbb{R}^{(n^k)},\quad F=(F_{i_1\cdots i_k}))\,.$$

In components

$$(Q(F))_{i_1\cdots i_k}=\rho_{i_1\cdots i_k}\circ L_X(\rho^{-1}\circ F\circ\varphi)\circ\varphi^{-1}=(L_X(\rho^{-1}\circ F\circ\varphi))_{i_1\cdots i_k}\circ\varphi^{-1}\,.$$

Recall that if T is any k-covariant tensor field on U then

$$(L_XT)_{i_1\cdots i_k}\circ\varphi^{-1}=\sum_{p=1}^n(X^p\circ\varphi^{-1})\frac{\partial(T_{i_1\cdots i_k}\circ\varphi^{-1})}{\partial x^p}+$$
$$\frac{\partial(X^p\circ\varphi^{-1})}{\partial x^{i_1}}(T_{pi_2\cdots i_k}\circ\varphi^{-1})+\cdots+\frac{\partial(X^p\circ\varphi^{-1})}{\partial x^{i_k}}(T_{i_1\cdots i_{k-1}p}\circ\varphi^{-1})\,.$$

Therefore,

$$(Q(F))_{i_1\cdots i_k}=\sum_{p=1}^n(X^p\circ\varphi^{-1})\frac{\partial F_{i_1\cdots i_k}}{\partial x^p}+$$
$$\frac{\partial(X^p\circ\varphi^{-1})}{\partial x^{i_1}}F_{pi_2\cdots i_k}+\cdots+\frac{\partial(X^p\circ\varphi^{-1})}{\partial x^{i_k}}F_{i_1\cdots i_{k-1}p}\,.$$

Now, note that

$$(Q(F))_{i_1\cdots i_k}=\sum_{j_1\cdots j_k=1}^n Q_{(i_1\cdots i_k)(j_1\cdots j_k)}(F_{j_1\cdots j_k})\,,$$

where

$$Q_{(i_1\cdots i_k)(j_1\cdots j_k)}:C^\infty(\varphi(U),\mathbb{R})\to\text{Func}(\varphi(U),\mathbb{R})\,,$$

and for all $f \in C^\infty(\varphi(U),\mathbb{R})$, $Q_{(i_1\cdots i_k)(j_1\cdots j_k)}(f)$ is the $(i_1\cdots i_k)$-component of $Q(F)$ with

$$F_{i_1\cdots i_k} = \begin{cases} f & \text{if } i_1 = j_1, \cdots, i_k = j_k \\ 0 & \text{otherwise} \end{cases} .$$

Hence

$$Q_{(i_1\cdots i_k)(j_1\cdots j_k)}(f) = \sum_{p=1}^{n} \delta^{i_1}_{j_1}\cdots\delta^{i_k}_{j_k}(X^p\circ\varphi^{-1})\frac{\partial f}{\partial x^p} + \\ \delta^{i_2}_{j_2}\cdots\delta^{i_k}_{j_k}\frac{\partial(X^{j_1}\circ\varphi^{-1})}{\partial x^{i_1}}f + \cdots + \delta^{i_1}_{j_1}\cdots\delta^{i_{k-1}}_{j_{k-1}}\frac{\partial(X^{j_1}\circ\varphi^{-1})}{\partial x^{i_k}}f\,.$$

Now, suppose $f \in C_c^\infty(\varphi(U))$. So, clearly, $f \in W^{e,q}_{loc}(\varphi(U))$. It follows from the hypotheses on e and q that (see Theorem 3.14)

$$W^{\tilde{s},\tilde{p}}_{loc}(\varphi(U)) \times W^{e-1,q}_{loc}(\varphi(U)) \hookrightarrow W^{e-1,q}_{loc}(\varphi(U))\,,$$
$$W^{\tilde{s}-1,\tilde{p}}_{loc}(\varphi(U)) \times W^{e,q}_{loc}(\varphi(U)) \hookrightarrow W^{e-1,q}_{loc}(\varphi(U))\,.$$

Also, by Corollary 3.21 and Theorem 3.11, we know that for all p and q, $X^p\circ\varphi^{-1}$ is in $W^{\tilde{s},\tilde{p}}_{loc}$ and $\frac{\partial(X^p\circ\varphi^{-1})}{\partial x^q}$ is in $W^{\tilde{s}-1,\tilde{p}}_{loc}$. Hence

$$Q_{(i_1\cdots i_k)(j_1\cdots j_k)} : (C_c^\infty(\varphi(U)), \|.\|_{e,q}) \to W^{e-1,q}_{loc}(\varphi(U))$$

is continuous.

Remark A.3. *Instead of A.3 and A.4, we may alternatively assume that for all balls $\Omega \subseteq \mathbb{R}^n$*

$$W^{\tilde{s},\tilde{p}}(\Omega) \times W^{e-1,q}(\Omega) \hookrightarrow W^{e-1,q}(\Omega)\,,$$
$$W^{\tilde{s}-1,\tilde{p}}(\Omega) \times W^{e,q}(\Omega) \hookrightarrow W^{e-1,q}(\Omega)\,,$$

and work with nice charts instead of super nice charts. However, if we do so, then we need to additionally assume that (see Theorem 3.9)

- *$\tilde{s}$ and $\tilde{p}$ are such that $\frac{\partial}{\partial x^j} : W^{\tilde{s},\tilde{p}}(\Omega) \to W^{\tilde{s}-1,\tilde{p}}(\Omega)$ ($1 \leq j \leq n$) is continuous.*
- *e and q are such that $\frac{\partial}{\partial x^j} : W^{e,q}(\Omega) \to W^{e-1,q}(\Omega)$ ($1 \leq j \leq n$) is continuous.*

- **Example 5: Covariant Derivative** Consider $\nabla : C^\infty(T^k_l M) \to \Gamma(T^{k+1}_l M)$. Here we will show that if e and q are such that

$$W^{s-1,p}(\mathbb{R}^n) \times W^{e,q}(\mathbb{R}^n) \hookrightarrow W^{e-1,q}(\mathbb{R}^n)\,, \tag{A.5}$$

then ∇ can be considered as a continuous operator from $W^{e,q}(T^k_l M)$ to $W^{e-1,q}(T^{k+1}_l M)$. The local representation of covariant derivative with respect to the coordinate chart (U,φ) is $Q : C^\infty(\varphi(U),\mathbb{R}^{n^{k+l}}) \to \text{Func}(\varphi(U),\mathbb{R}^{n^{k+l+1}})$ defined by

$$Q(F) = \tilde{\rho}\circ\nabla(\rho^{-1}\circ F\circ\varphi)\circ\varphi^{-1} \quad (F : \varphi(U) \to \mathbb{R}^{(n^{k+l})}, \quad F = (F^{j_1\cdots j_l}_{i_1\cdots i_k}))\,.$$

In components

$$(Q(F))^{j_1\cdots j_l}_{i_1\cdots i_k r} = \tilde{\rho}^{j_1\cdots j_l}_{i_1\cdots i_k r}\circ\nabla(\rho^{-1}\circ F\circ\varphi)\circ\varphi^{-1} = (\nabla(\rho^{-1}\circ F\circ\varphi))^{j_1\cdots j_l}_{i_1\cdots i_k r}\circ\varphi^{-1}\,.$$

Recall that if T is any $\binom{k}{l}$-covariant tensor field on U then

$$
\begin{aligned}
&(\nabla T)^{j_1\cdots j_l}_{i_1\cdots i_k r}\circ\varphi^{-1}=(\nabla_r T)^{j_1\cdots j_l}_{i_1\cdots i_k}\circ\varphi^{-1}\\
&\quad=\frac{\partial}{\partial x^r}(T^{j_1\cdots j_l}_{i_1\cdots i_k}\circ\varphi^{-1})+\sum_{p=1}^{n}(T^{pj_2\cdots j_l}_{i_1\cdots i_k}\circ\varphi^{-1})(\Gamma^{j_1}_{rp}\circ\varphi^{-1})+\cdots+(T^{j_1\cdots j_{l-1}p}_{i_1\cdots i_k}\circ\varphi^{-1})(\Gamma^{j_l}_{rp}\circ\varphi^{-1})\\
&\quad-\sum_{p=1}^{n}(T^{j_1\cdots j_l}_{pi_2\cdots i_k}\circ\varphi^{-1})(\Gamma^{p}_{ri_1}\circ\varphi^{-1})+\cdots+(T^{j_1\cdots j_l}_{i_1\cdots i_{k-1}p}\circ\varphi^{-1})(\Gamma^{p}_{ri_k}\circ\varphi^{-1})\,.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(Q(F))^{j_1\cdots j_l}_{i_1\cdots i_k r}&=\frac{\partial}{\partial x^r}F^{j_1\cdots j_l}_{i_1\cdots i_k}\\
&\quad+\sum_{p=1}^{n}(F^{pj_2\cdots j_l}_{i_1\cdots i_k})(\Gamma^{j_1}_{rp}\circ\varphi^{-1})+\cdots+(F^{j_1\cdots j_{l-1}p}_{i_1\cdots i_k})(\Gamma^{j_l}_{rp}\circ\varphi^{-1})\\
&\quad-\sum_{p=1}^{n}(F^{j_1\cdots j_l}_{pi_2\cdots i_k})(\Gamma^{p}_{ri_1}\circ\varphi^{-1})+\cdots+(F^{j_1\cdots j_l}_{i_1\cdots i_{k-1}p})(\Gamma^{p}_{ri_k}\circ\varphi^{-1})\,.
\end{aligned}
$$

Now, note that

$$
(Q(F))^{j_1\cdots j_l}_{i_1\cdots i_k r}=\sum_{\hat{j}_1,\cdots,\hat{j}_l,\hat{i}_1,\cdots,\hat{i}_k=1}^{n}Q^{(j_1\cdots j_l)(\hat{j}_1\cdots\hat{j}_l)}_{(i_1\cdots i_k r)(\hat{i}_1\cdots\hat{i}_k)}(F^{\hat{j}_1\cdots\hat{j}_l}_{\hat{i}_1\cdots\hat{i}_k})\,,
$$

where

$$
Q^{(j_1\cdots j_l)(\hat{j}_1\cdots\hat{j}_l)}_{(i_1\cdots i_k r)(\hat{i}_1\cdots\hat{i}_k)}:C^\infty(\varphi(U),\mathbb{R})\to \text{Func}(\varphi(U),\mathbb{R})\,,
$$

and for all $f\in C^\infty(\varphi(U),\mathbb{R})$, $Q^{(j_1\cdots j_l)(\hat{j}_1\cdots\hat{j}_l)}_{(i_1\cdots i_k r)(\hat{i}_1\cdots\hat{i}_k)}(f)$ is the $(i_1\cdots i_k r)^{(j_1\cdots j_l)}$-component of $Q(F)$ with

$$
F^{j_1\cdots j_l}_{i_1\cdots i_k}=\begin{cases} f & \text{if } i_1=\hat{i}_1,\cdots,i_k=\hat{i}_k, j_1=\hat{j}_1,\cdots,j_l=\hat{j}_l\\ 0 & \text{otherwise}\end{cases}\,.
$$

Hence

$$
\begin{aligned}
&Q^{(j_1\cdots j_l)(\hat{j}_1\cdots\hat{j}_l)}_{(i_1\cdots i_k r)(\hat{i}_1\cdots\hat{i}_k)}(f)=\delta^{i_1}_{\hat{i}_1}\cdots\delta^{i_k}_{\hat{i}_k}\delta^{j_1}_{\hat{j}_1}\cdots\delta^{j_l}_{\hat{j}_l}\frac{\partial}{\partial x^r}f\\
&+\delta^{i_1}_{\hat{i}_1}\cdots\delta^{i_k}_{\hat{i}_k}\delta^{j_2}_{\hat{j}_2}\cdots\delta^{j_l}_{\hat{j}_l}(f)(\Gamma^{j_1}_{r\hat{j}_1}\circ\varphi^{-1})+\cdots+\delta^{i_1}_{\hat{i}_1}\cdots\delta^{i_k}_{\hat{i}_k}\delta^{j_1}_{\hat{j}_1}\cdots\delta^{j_{l-1}}_{\hat{j}_{l-1}}(f)(\Gamma^{j_l}_{r\hat{j}_l}\circ\varphi^{-1})\\
&-\delta^{i_2}_{\hat{i}_2}\cdots\delta^{i_k}_{\hat{i}_k}\delta^{j_1}_{\hat{j}_1}\cdots\delta^{j_l}_{\hat{j}_l}(f)(\Gamma^{\hat{i}_1}_{ri_1}\circ\varphi^{-1})+\cdots+\delta^{i_1}_{\hat{i}_1}\cdots\delta^{i_{k-1}}_{\hat{i}_{k-1}}\delta^{j_1}_{\hat{j}_1}\cdots\delta^{j_l}_{\hat{j}_l}(f)(\Gamma^{\hat{i}_k}_{ri_k}\circ\varphi^{-1})\,.
\end{aligned}
$$

Now, suppose $f\in C_c^\infty(\varphi(U))$. So, clearly, $f\in W^{e,q}_{loc}(\varphi(U))$. It follows from the hypotheses on e and q that (see Theorem 3.14)

$$
W^{s-1,p}_{loc}(\varphi(U))\times W^{e,q}_{loc}(\varphi(U))\hookrightarrow W^{e-1,q}_{loc}(\varphi(U))\,.
$$

Also, we know that for all a, b, and c, $\Gamma^a_{bc}\circ\varphi^{-1}$ is in $W^{s-1,p}_{loc}$. Hence

$$
Q^{(j_1\cdots j_l)(\hat{j}_1\cdots\hat{j}_l)}_{(i_1\cdots i_k r)(\hat{i}_1\cdots\hat{i}_k)}:(C_c^\infty(\varphi(U)),\|.\|_{e,q})\to W^{e-1,q}_{loc}(\varphi(U))
$$

is continuous.

Remark A.4. *Instead of A.5, we may alternatively assume that for all balls $\Omega\subseteq\mathbb{R}^n$*

$$
W^{s-1,p}(\Omega)\times W^{e,q}(\Omega)\hookrightarrow W^{e-1,q}(\Omega)\,,
$$

and work with nice charts instead of super nice charts. However, if we do so, then we need to additionally assume that e *and* q *are such that* $\frac{\partial}{\partial x^j} : W^{e,q}(\Omega) \to W^{e-1,q}(\Omega)$ *(*$1 \leq j \leq n$*) is continuous (see Theorem 3.9).*

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