OPTIMALITY OF MULTILEVEL PRECONDITIONERS FOR LOCAL MESH REFINEMENT IN THREE DIMENSIONS

BURAK AKSOYLU AND MICHAEL HOLST

ABSTRACT. In this article, we establish optimality of the Bramble-Pasciak-Xu (BPX) norm equivalence and optimality of the wavelet modified (or *stabilized*) hierarchical basis (WHB) preconditioner in the setting of local 3D mesh refinement. In the analysis of WHB methods, a critical first step is to establish the optimality of BPX norm equivalence for the refinement procedures under consideration. While the available optimality results for the BPX norm have been constructed primarily in the setting of uniformly refined meshes, a notable exception is the local 2D red-green result due to Dahmen and Kunoth. The purpose of this article is to extend this original 2D optimality result to the local 3D red-green refinement procedure introduced by Bornemann-Erdmann-Kornhuber (BEK), and then to use this result to extend the WHB optimality results from the quasiuniform setting to local 2D and 3D red-green refinement scenarios. The BPX extension is reduced to establishing that locally enriched finite element subspaces allow for the construction of a scaled basis which is formally Riesz stable. This construction turns out to rest not only on shape regularity of the refined elements, but also critically on a number of geometrical properties we establish between neighboring simplices produced by the BEK refinement procedure. It is possible to show that the number of degrees of freedom used for smoothing is bounded by a constant times the number of degrees of freedom introduced at that level of refinement, indicating that a practical implementable version of the resulting BPX preconditioner for the BEK refinement setting has provably optimal (linear) computational complexity per iteration. An interesting implication of the optimality of the WHB preconditioner is the *a priori* H^1-stability of the L_2-projection. The existing *a posteriori* approaches in the literature dictate a reconstruction of the mesh if such conditions cannot be satisfied. The theoretical framework employed supports arbitrary spatial dimension $d \geq 1$ and requires no coefficient smoothness assumptions beyond those required for well-posedness in H^1.

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1. INTRODUCTION

In this article, we analyze the impact of local mesh refinement on the stability of multilevel finite element spaces and on optimality (linear space and time complexity) of multilevel preconditioners. Adaptive refinement techniques have become a crucial tool for many applications, and access to optimal or near-optimal multilevel preconditioners for locally refined mesh situations is of primary concern to computational scientists. The preconditioners which can be expected to have somewhat favorable space and time complexity in such local refinement scenarios are the hierarchical basis (HB) method [9], the Bramble-Pasciak-Xu (BPX) preconditioner [16], and the wavelet modified (or stabilized) hierarchical basis (WHB) method [35]. While there are optimality results for both the BPX and WHB preconditioners in the literature, these are primarily for quasiuniform meshes and/or two space dimensions (with some exceptions noted below). In particular, there are few hard results in the literature on the optimality of these methods for various realistic local mesh refinement hierarchies, especially in three space dimensions. In this article, the first in a series of two articles [2] on local refinement and multilevel preconditioners, we first assemble optimality results for the BPX norm equivalence in local refinement scenarios in three spacial dimensions. Building on the extended BPX results, we then develop optimality results for the WHB method in local refinement settings. The material forming this series is based on the first author's Ph.D. dissertation [1] and comprehensive presentation of this article can be found in [3, 4, 5, 6].

Through some topological or geometrical abstraction, if local refinement is extended to d spatial dimensions, then the main results are valid for any dimension $d \geq 1$ and for nonsmooth PDE coefficients $p \in L_\infty(\Omega)$. Throughout this article, we consider primarily the $d = 3$ case. But, when the abstraction to generic d is clear, we simply state the argument by using this generic d.

The problem class we focus on here is linear second order partial differential equations (PDE) of the form:

$$-\nabla \cdot (p\,\nabla u) + q\,u = f,\ u = 0 \text{ on } \partial\Omega. \tag{1.1}$$

Here, $f \in L_2(\Omega)$, $p, q \in L_\infty(\Omega)$, $p : \Omega \rightarrow L(\Re^d, \Re^d)$, $q : \Omega \rightarrow \Re$, where p is a symmetric positive definite matrix function, and where q is a nonnegative function. Let $\mathcal{T}_0$ be a shape regular and quasiuniform initial partition of Ω into a finite number of d simplices, and generate $\mathcal{T}_1, \mathcal{T}_2, \ldots$ by refining the initial partition using red-green local refinement strategies in $d = 3$ spatial dimensions. Denote as $\mathcal{S}_j$ the simplicial linear C^0 finite element space corresponding to $\mathcal{T}_j$ equipped with zero boundary values. The set of nodal basis functions for $\mathcal{S}_j$ is denoted by $\Phi^{(j)} = \{\phi_i^{(j)}\}_{i=1}^{N_j}$ where $N_j = \dim \mathcal{S}_j$ is equal to the number of interior nodes in $\mathcal{T}_j$, representing the number of degrees of freedom in the discrete space. Successively refined finite element spaces will form the following nested sequence:

$$\mathcal{S}_0 \subset \mathcal{S}_1 \subset \ldots \subset \mathcal{S}_j \subset \ldots \subset H_0^1(\Omega).$$

Let the bilinear form and the functional associated with the weak formulation of (1.1) be denoted as

$$a(u, v) = \int_\Omega p\,\nabla u \cdot \nabla v + q\,u\,v\,dx,\ b(v) = \int_\Omega f\,v\,dx,\ u, v \in H_0^1(\Omega).$$

We consider primarily the following Galerkin formulation: Find $u \in \mathcal{S}_j$, such that

$$a(u, v) = b(v),\ \forall v \in \mathcal{S}_j. \tag{1.2}$$

The finite element approximation in $\mathcal{S}_j$ has the form $u^{(j)} = \sum_{i=1}^{N_j} u_i \phi_i^{(j)}$, where $u = (u_1, \ldots, u_{N_j})^T$ denotes the coefficients of $u^{(j)}$ with respect to $\Phi^{(j)}$. The resulting *discretization operator* $A^{(j)} = \{a(\phi_k^{(j)}, \phi_l^{(j)})\}_{k,l=1}^{N_j}$ must be inverted numerically to determine the coefficients u from the linear system:

$$A^{(j)} u = F^{(j)}, \tag{1.3}$$

where $F^{(j)} = \{b(\phi_l^{(j)})\}_{l=1}^{N_j}$. Our task is to solve (1.3) with optimal (linear) complexity in both storage and computation, where the finite element spaces $\mathcal{S}_j$ are built on locally refined meshes.

Optimality of the BPX norm equivalence with generic local refinement was shown by Bramble and Pasciak [14], where the impact of the local smoother and the local projection operator on the estimates was carefully analyzed. The two primary results on optimality of the BPX norm equivalence in the local refinement settings are due to Dahmen and Kunoth [19] and Bornemann and Yserentant [12]. Both works consider only two space dimensions, and in particular, the refinement strategies analyzed are restricted 2D red-green refinement and 2D red refinement, respectively. In this paper, we extend the framework developed in [19] to a practical, implementable 3D local red-green refinement procedure introduced by Bornemann-Erdmann-Kornhuber (BEK) [11]. We will refer to this as the BEK refinement procedure.

HB methods [9, 7, 37] are particularly attractive in the local refinement setting because (by construction) each iteration has linear (optimal) computational and storage complexity. Unfortunately, the resulting preconditioner is not optimal due to condition number growth: in two dimensions the growth is slow, and the method is quite effective (nearly optimal), but in three dimensions the condition number grows much more rapidly with the number of unknowns [26]. To address this instability, one can employ L_2-orthonormal wavelets in place of the hierarchical basis giving rise to an optimal preconditioner [23]. However, the complicated nature of traditional wavelet bases, in particular the non-local support of the basis functions and problematic treatment of boundary conditions, severely limits computational feasibility. WHB methods have been developed [34, 35] as an alternative, and they can be interpreted as a wavelet modification (or *stabilization*) of the hierarchical basis. These methods have been shown to optimally stabilize the condition number of the systems arising from hierarchical basis methods on quasiuniform meshes in both two and three space dimensions, and retain a comparable cost per iteration.

There are two main results and one side result in this article. The main results establish the optimality of the BPX norm equivalence and also optimality of the WHB preconditoner—as well as optimal computational complexity per iteration—for the resulting locally refined 3D finite element hierarchy. Both the BPX and WHB preconditioners under consideration are additive Schwarz preconditioners. The BPX analysis here heavily relies on the techniques of the Dahmen-Kunoth [19] framework and can be seen as an extension to three spatial dimensions with the realistic BEK refinement procedure [11] being the application of interest. The WHB framework relies on the optimality of the BPX norm equivalence. Hence, the WHB results are established after the BPX results.

The side result is the H^1-stability of L_2-projection onto finite element spaces built through the BEK local refinement procedure. This question is currently under intensive study in the finite element community due to its relationship to multilevel preconditioning. The existing theoretical results, due primarily to Carstensen [18] and Bramble-Pasciak-Steinbach [15] involve *a posteriori* verification of somewhat complicated mesh conditions after local refinement has taken place. If such mesh conditions are not satisfied, one has to redefine the mesh. However, an interesting consequence of the BPX optimality results for locally refined 2D and 3D meshes established here is H^1-stability of L_2-projection restricted to the same locally enriched finite element spaces. This result appears to be the first *a priori* H^1-stability result for L_2-projection on finite element spaces produced by practical and easily implementable 2D and 3D local refinement procedures.

Outline of the paper. In §2, we introduce some basic approximation theory tools used in the analysis such as Besov spaces and Bernstein inequalities. The framework for the main norm equivalence is also established here. In §3, we list the BEK refinement conditions. We give several theorems about the generation and size relations of the neighboring simplices, thereby establishing local (patchwise) quasiuniformity. This gives rise to an L_2-stable Riesz basis in §3.1; one can then establish the Bernstein inequality. In §4, we explicitly give an upper bound for the nodes introduced in the refinement region. This implies that one application of the BPX preconditioner to a function has linear (optimal) computational complexity. In §5, we use the geometrical results from §3 to extend the 2D Dahmen-Kunoth results to the 3D BEK refinement procedure by establishing the desired norm equivalence. While it is not possible to establish a Jackson inequality due to the nature of local adaptivity, in §6 the remaining inequality in the norm equivalence is handled directly using approximation theory tools, as in the original work [19]. In §7, we introduce the WHB preconditioner as well as the operator used in its definition. In §8, we state the fundamental assumption for establishing basis stability and set up the main theoretical results for the WHB framework, namely, optimality of the WHB preconditioner in the 2D and 3D local red-green refinements. The results in §8 rest completely on the BPX results in §5 and on the Bernstein inequalities, the latter of which rest on the geometrical results established in §3. The first *a priori* H^1-stability result for L_2-projection on the finite element spaces produced is established in §9. We conclude in §10.

2. PRELIMINARIES AND THE MAIN NORM EQUIVALENCE

The basic restriction on the refinement procedure is that it remains *nested*. In other words, tetrahedra of level j which are not candidates for further refinement will never be touched in the future. Let Ω_j denote the refinement region, namely, the union of the supports of basis functions which are introduced at level j. Due to nested refinement $\Omega_j \subset \Omega_{j-1}$. Then the following hierarchy holds:

$$\Omega_J \subset \Omega_{J-1} \subset \cdots \subset \Omega_0 = \Omega. \tag{2.1}$$

In the local refinement setting, in order to maintain optimal computational complexity, the smoother is restricted to a local space $\tilde{\mathcal{S}}_j$, typically

$$\mathcal{S}_j^f \subseteq \tilde{\mathcal{S}}_j \subset \mathcal{S}_j, \tag{2.2}$$

where $\mathcal{S}_j^f := (I_j - I_{j-1})\, \mathcal{S}_j$ and $I_j : L_2(\Omega) \to \mathcal{S}_j$ denotes the finite element interpolation operator. Degrees of freedom (DOF) corresponding to $\mathcal{S}_j^f$ and $\tilde{\mathcal{S}}_j$ will be denoted by $\mathcal{N}_j^f$ and $\tilde{\mathcal{N}}_j$ respectively where f stands for *fine*. (2.2) indicates that $\mathcal{N}_j^f \subseteq \tilde{\mathcal{N}}_j$, typically, $\tilde{\mathcal{N}}_j$ consists of fine DOF and their corresponding coarse fathers.

The BPX preconditioner (also known as parallelized or additive multigrid) is defined as follows:

$$Xu := \sum_{j=0}^{J} 2^{j(d-2)} \sum_{i \in \tilde{\mathcal{N}}_j} (u, \phi_i^{(j)}) \phi_i^{(j)}. \tag{2.3}$$

Success of the BPX preconditioner in locally refined regimes relies on the fact the BPX smoother acts on a local space as in (2.2). As mentioned above, it acts on a slightly bigger set than fine DOF (examples of these are given in [13]). Choice of such a set is crucial because computational cost per iteration will eventually determine the overall computational complexity of the method. Hence in §4, we show that the overall computational cost of the smoother is $O(N)$, meaning that the BPX preconditioner is optimal per iteration. We would like to emphasize that one of the the main goals of this paper, as in the earlier works of Dahmen-Kunoth [19] and Bornemann-Yserentant [12] in the purely two-dimensional case, is to establish the optimality of the BPX norm equivalence:

$$c_1 \sum_{j=0}^{J} 2^{2j} \|(Q_j - Q_{j-1})u\|_{L_2}^2 \leq \|u\|_{H^1}^2 \leq c_2 \sum_{j=0}^{J} 2^{2j} \|(Q_j - Q_{j-1})u\|_{L_2}^2, \qquad (2.4)$$

where Q_j is the L_2-projection. We note that in the uniform refinement setting, it is straight-forward to link the BPX norm equivalence to the optimality of the BPX preconditioner:

$$c_1(Xu, u) \leq \|u\|_{H^1}^2 \leq c_2(Xu, u),$$

due to the projector relationships between the Q_j operators. However, in the local refinement scenario the precise link between the norm equivalence and the preconditioner is more subtle and remains essentially open.

The rest of this section is dedicated to setting up the framework to establish the main norm equivalence (2.4) which will be formalized in Theorem 2.1 at the end of this section. We borrow several tools from approximation theory, including the modulus of smoothness, $\omega_k(f, t, \Omega)_p$, which is a finer scale of smoothness than differentiability. It is a central tool in the analysis here and it naturally gives rise to the notion of *Besov spaces*. For further details and definitions, see [19, 29]. Besov spaces are defined to be the collection of functions $f \in L_p(\Omega)$ with a finite Besov norm defined as follows:

$$\|f\|_{B_{p,q}^s(\Omega)}^q := \|f\|_{L_p(\Omega)}^q + |f|_{B_{p,q}^s(\Omega)}^q,$$

where the seminorm is given by

$$|f|_{B_{p,q}^s(\Omega)} := \|\{2^{sj}\omega_k(f, 2^{-j}, \Omega)_p\}_{j \in \mathbf{b}N_0}\|_{l_q},$$

with k any fixed integer larger than s.

Besov spaces become the primary function space setting in the analysis by realizing Sobolev spaces as Besov spaces:

$$H^s(\Omega) \cong B_{2,2}^s(\Omega), \ s > 0.$$

The primary motivation for employing the Besov space stems from the fact that the characterization of functions which have a given upper bound for the error of approximation sometimes calls for a finer scale of smoothness that provided by Sobolev classes functions.

The Bernstein inequality is defined as:

$$\omega_{k+1}(u, t)_p \leq c\,(\min\{1, t2^J\})^{\beta} \|u\|_{L_p}, \ u \in \mathcal{S}_j, \ j = 0, \ldots, J, \qquad (2.5)$$

where c is independent of u and j. Usually $k =$ degree of the element and in the case of linear finite elements $k = 1$. Here β is determined by the global smoothness of the approximation space as well as p. For C^r finite elements, $\beta = \min\{1 + r + \frac{1}{p}, k + 1\}$.

Let θ_J be defined as follows.

$$\theta_{j,J} := \sup_{u \in \mathcal{S}_J} \frac{\|u - Q_j u\|_{L_2}}{\omega_2(u, 2^{-j})_2}, \ \theta_J := \max\{1, \theta_{j,J} : j = 0, \ldots, J\}. \qquad (2.6)$$

Following [19] we have then

Theorem 2.1. *Suppose the Bernstein inequality (2.5) holds for some real number $\beta > 1$. Then, for each $0 < s < \min\{\beta, 2\}$, there exist constants $0 < c_1,\ c_2 < \infty$ independent of $u \in \mathcal{S}_J,\ J = 0, 1, \ldots$, such that the following norm equivalence holds:*

$$\frac{c_1}{\theta_J^2} \sum_{j=0}^{J} 2^{2j} \|(Q_j - Q_{j-1})u\|_{L_2}^2 \leq \|u\|_{H^1}^2 \leq c_2 \sum_{j=0}^{J} 2^{2j} \|(Q_j - Q_{j-1})u\|_{L_2}^2,\ u \in \mathcal{S}_J. \quad (2.7)$$

Proof. See [19, Theorem 4.1]. □

We would like to elaborate on the difficulities one faces within the local refinement framework. In order Bernstein inequality to hold, one needs to establish that the underlying basis is L_2-stable Riesz basis as in (3.8). This crucial property heavily depends on local quasiuniformity of the mesh. Hence, Bernstein inequality is established in §5 through local quasiuniformity and L_2-stability of the basis in the Riesz sense.

A Jackson-type inequality cannot hold in a local refinement setting. This poses a major difficulty in the analysis because one has to calculate θ_J directly. The missing crucial piece of the optimal norm equivalence in (2.7), namely, $\theta_J = O(1)$ as $J \to \infty$, will be shown in (6.12) so that (2.4) holds. This required the operator $\tilde{Q}_j$ to be bounded locally and to fix polynomials of degree 1 as will be shown in §6.

3. THE BEK REFINEMENT PROCEDURE

Our interest is to show optimality of the BPX norm equivalence for the local 3D red-green refinement introduced by Bornemann-Erdmann-Kornhuber [11]. This 3D red-green refinement is practical, easy to implement, and numerical experiments were presented in [11]. A similar refinement procedure was analyzed by Bey [10]; in particular, the same green closure strategy was used in both papers. While these refinement procedures are known to be asymptotically non-degenerate (and thus produce shape regular simplices at every level of refinement), shape regularity is insufficient to construct a stable Riesz basis for finite element spaces on locally adapted meshes. To construct a stable Riesz basis we will need to establish patchwise quasiuniformity as in [19]; as a result, d-vertex adjacency relationships that are independent of shape regularity of the elements must be established between neighboring tetrahedra as done in [19] for triangles.

We first list a number of geometric assumptions we make concerning the underlying mesh. Let $\Omega \subset \Re^3$ be a polyhedral domain. We assume that the triangulation $\mathcal{T}_j$ of Ω at level j is a collection of tetrahedra with mutually disjoint interiors which cover $\Omega = \bigcup_{\tau \in \mathcal{T}_j} \tau$. We want to generate successive refinements $\mathcal{T}_0, \mathcal{T}_1, \ldots$ which satisfy the following conditions:

Assumption 3.1. Nestedness: *Each tetrahedron (son) $\tau \in \mathcal{T}_j$ is covered by exactly one tetrahedron (father) $\tau' \in \mathcal{T}_{j-1}$, and any corner of τ is either a corner or an edge midpoint of τ'.*

Assumption 3.2. Conformity: *The intersection of any two tetrahedra $\tau, \tau' \in \mathcal{T}_j$ is either empty, a common vertex, a common edge or a common face.*

Assumption 3.3. Nondegeneracy: *The interior angles of all tetrahedra in the refinement sequence $\mathcal{T}_0, \mathcal{T}_1, \ldots$ are bounded away from zero.*

A regular (red) refinement subdivides a tetrahedron τ into 8 equal volume subtetrahedra. We connect the edges of each face as in 2D regular refinement. We then cut off four

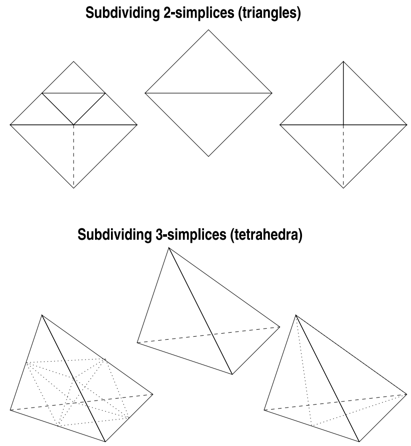

FIGURE 1.

subtetrahedra at the corners which are congruent to τ. An octahedron with three parallelograms remains in the interior. Cutting the octahedron along the two faces of these parallelograms, we obtain four more subtetrahedra which are not necessarily congruent to τ. We choose the diagonal of the parallelogram so that the successive refinements always preserve nondegeneracy [1, 10, 27, 38]. A sketch of regular refinement (octasection and quadrasection in 3D and 2D, respectively) as well as bisection is given in Figure 1.

If a tetrahedron is marked for regular refinement, the resulting triangulation violates conformity A.3.2. Nonconformity is then remedied by irregular (green) refinement. In 3D, there are altogether $2^6 = 64$ possible edge refinements, of which 62 are irregular. One must pay extra attention to irregular refinement in the implementation due to the large number of possible nonconforming configurations. Bey [10] gives a methodical way of handling irregular cases. Using symmetry arguments, the 62 irregular cases can be divided into 9 different types. To ensure that the interior angles remain bounded away from zero, we enforce the following additional conditions. (Identical assumptions were made in [19] for their 2D refinement analogue.)

Assumption 3.4. *Irregular tetrahedra are not refined further.*

Assumption 3.5. *Only tetrahedra $\tau \in \mathcal{T}_j$ with $L(\tau) = j$ are refined for the construction of $\mathcal{T}_{j+1}$, where $L(\tau) = \min\{j : \tau \in \mathcal{T}_j\}$ denotes the level of τ.*

One should note that the restrictive character of A.3.4 and A.3.5 can be eliminated by a modification on the sequence of the tetrahedralizations [10]. On the other hand, it is straightforward to enforce both assumptions in a typical local refinement algorithm by minor modifications of the supporting datastructures for tetrahedral elements (cf. [22]). In any event, the proof technique (see (6.8) and (6.9)) requires both assumptions hold. The last refinement condition enforced for the possible 62 irregularly refined tetrahedra is stated as the following.

Assumption 3.6. *If three or more edges are refined and do not belong to a common face, then the tetrahedron is refined regularly.*

We note that the d-vertex adjacency generation bound for simplices in $\Re^d$ which are adjacent at d vertices *is the primary result required in the support of a basis function so that (3.6) holds, and depends delicately on the particular details of the local refinement procedure rather than on shape regularity of the elements.* The generation bound for simplices which are adjacent at $d-1, d-2, \ldots$ vertices follows by using the shape regularity and the generation bound established for d-vertex adjacency. We provide rigorous generation bounds for all the adjacency types mentioned in the lemmas to follow when $d = 3$. The 2D version appeared in [19]; the 3D extension is as described below.

Lemma 3.7. *Let τ and τ' be two tetrahedra in $\mathcal{T}_j$ sharing a common face f. Then*

$$|L(\tau) - L(\tau')| \leq 1. \tag{3.1}$$

Proof. If $L(\tau) = L(\tau')$, then $0 \leq 1$, there is nothing to show. Without loss of generality, assume that $L(\tau) < L(\tau')$. Proof requires a detailed and systematic analysis. To show the line of reasoning, we first list the facts used in the proof:

1. $L(\tau') \leq j$ because by assumption $\tau' \in \mathcal{T}_j$. Then, $L(\tau) < j$.
2. By assumption $\tau \in \mathcal{T}_j$, meaning that τ was never refined from the level it was born $L(\tau)$ to level j.
3. Let τ'' be the father of τ'. Then $L(\tau'') = L(\tau') - 1 < j$.
4. $L(\tau) < L(\tau')$ by assumption, implying $L(\tau) \leq L(\tau'')$.
5. By (2), τ belongs to all the triangulations from $L(\tau)$ to j, in particular $\tau \in \mathcal{T}_{L(\tau'')}$, where by (3) $L(\tau'') < j$.

f is the common face of τ and τ' on level j. By (5) both $\tau,\ \tau'' \in \mathcal{T}_{L(\tau'')}$. Then, A.3.2 implies that f must still be the common face of τ and τ''. Hence, τ' must have been irregular.

On the other hand, $L(\tau) \leq L(\tau') - 1 = L(\tau'')$. Next, we proceed by eliminating the possibility that $L(\tau) < L(\tau'')$. If so, we repeat the above reasoning, and τ'' becomes irregular. τ'' is already the father of the irregular τ', contradicting A.3.4 for level $L(\tau'')$. Hence $L(\tau) = L(\tau'') = L(\tau') - 1$ concludes the proof. □

By A.3.4 and A.3.5, every tetrahedron at any $\mathcal{T}_j$ is geometrically similar to some tetrahedron in $\mathcal{T}_0$ or to a tetrahedron arising from an irregular refinement of some tetrahedron in $\mathcal{T}_0$. Then, there exist absolute constants $c_1,\ c_2$ such that

$$c_1 \operatorname{diam}(\bar{\tau})\, 2^{-L(\tau)} \leq \operatorname{diam}(\tau) \leq c_2 \operatorname{diam}(\bar{\tau})\, 2^{-L(\tau)}, \tag{3.2}$$

where $\bar{\tau}$ is the father of τ in the initial mesh. The lemma below follows by shape regularity and (3.1).

Lemma 3.8. *Let τ, τ' and ζ, ζ' be the tetrahedra in $\mathcal{T}_j$ sharing a common edge (two vertices) and a common vertex, respectively. Then there exist finite numbers V and E depending on the shape regularity such that*

$$|L(\tau) - L(\tau')| \leq V, \tag{3.3}$$

$$|L(\zeta) - L(\zeta')| \leq E. \tag{3.4}$$

Consequently, simplices in the support of a basis function are comparable in size as indicated in (3.5). This is usually called *patchwise quasiuniformity*. Furthermore, it was shown in [1] that patchwise quasiuniformity (3.5) holds for 3D marked tetrahedron bisection by Joe and Liu [24] and for 2D newest vertex bisection by Sewell [30] and Mitchell [25]. Due to the restrictive nature of the proof technique (see (6.8) and (6.9)), we focus on refinement procedures which obey A.3.4 and A.3.5. However, due to the strong geometrical results available for purely bisection-based local refinement procedures, it should be possible to establish the main results of this paper for purely bisection-based strategies.

Lemma 3.9. *There is a constant depending on the shape regularity of $\mathcal{T}_j$ and the quasi-uniformity of $\mathcal{T}_0$, such that*

$$\frac{\operatorname{diam}(\tau)}{\operatorname{diam}(\tau')} \leq c,\ \forall \tau, \tau' \in \mathcal{T}_j,\ \tau \cap \tau' \neq \emptyset. \tag{3.5}$$

Proof. τ and τ' are either face-adjacent (d vertices), edge-adjacent ($d-1$ vertices), or vertex-adjacent, and are handled by (3.1), (3.4), (3.3), respectively.

$$\begin{aligned} \frac{\operatorname{diam}(\tau)}{\operatorname{diam}(\tau')} &\leq c\, 2^{|L(\tau)-L(\tau')|}\, \frac{\operatorname{diam}(\bar{\tau})}{\operatorname{diam}(\bar{\tau}')} \text{ (by (3.2))} \\ &\leq c\, 2^{\max\{1,E,V\}}\, \gamma^{(0)} \text{ (by (3.1), (3.4), (3.3) and quasiuniformity of } \mathcal{T}_0) \end{aligned}$$

□

3.1. **L_2-stable Riesz basis.** Since patchwise quasiuniformity is established by (3.5), we can now take the first step in establishing the norm equivalence in section 5. In other words, our motivation is to form a stable basis in the following sense [29].

$$\| \sum_{x_i \in \mathcal{N}_j} u_i \phi_i^{(j)} \|_{L_2(\Omega)} \eqsim \| \{\text{volume}^{1/2}(\text{supp } \phi_i^{(j)})\ u_i\}_{x_i \in \mathcal{N}_j} \|_{l_2}. \tag{3.6}$$

The basis stability (3.6) will then guarantee that the Bernstein inequality (2.5) holds. For a stable basis, functions with small supports have to be augmented by an appropriate scaling so that $\|\phi_i^{(j)}\|_{L_2(\Omega)}$ remains roughly the same for all basis functions. This is reflected in $\text{volume}(\text{supp } \phi_i^{(j)})$ by defining:

$$L_{j,i} = \min\{L(\tau) : \tau \in \mathcal{T}_j,\ x_i \in \tau\}. \tag{3.7}$$

Then

$$\text{volume}(\text{supp } \phi_i^{(j)}) \eqsim 2^{-dL_{j,i}}.$$

We prefer to use an equivalent notion of basis stability; a basis is called L_2*-stable Riesz basis* if:

$$\| \sum_{x_i \in \mathcal{N}_j} \hat{u}_i \hat{\phi}_i^{(j)} \|_{L_2(\Omega)} \eqsim \| \{\hat{u}_i\}_{x_i \in \mathcal{N}_j} \|_{l_2}, \tag{3.8}$$

where $\hat{\phi}_i^{(j)}$ denotes the scaled basis, and the relationship between (3.6) and (3.8) is given as follows:

$$\hat{\phi}_i^{(j)} = 2^{d/2 L_{j,i}}\ \phi_i^{(j)},\ \hat{u}_i = 2^{-d/2 L_{j,i}}\ u_i,\ x_i \in \mathcal{N}_j. \tag{3.9}$$

Then (3.8) forms the sufficient condition to establish the Bernstein inequality (2.5). This crucial property helps us to prove Theorem 8.2.

Remark 3.10. *The analysis is done purely with basis functions, completely independent of the underlying mesh geometry. Furthermore, our construction works for any d-dimensional setting with the scaling (3.9). However, it is not clear how to define face-adjacency relations for $d > 3$. If such relations can be defined through some topological or geometrical abstraction, then our framework naturally extends to d-dimensional local refinement strategies, and hence the optimality of the BPX and WHB preconditioners can be guaranteed in $\Re^d$, $d \geq 1$. One such generalization was given by Brandts-Korotov-Krizek in* [17] *and in the references therein.*

4. Local smoothing computational complexity

In [11], the smoother is chosen to act on the local space

$$\tilde{\mathcal{S}}_j = \text{span}\left[\bigcup \{\phi_i^{(j)}\}_{i=N_{j-1}+1}^{N_j} \bigcup \{\phi_i^{(j)} \neq \phi_i^{(j-1)}\}_{i=1}^{N_{j-1}} \right].$$

Other choices for $\tilde{\mathcal{N}}_j$ are also possible; e.g., DOF which intersect the refinement region Ω_j [2, 14]. The only restriction is that $\tilde{\mathcal{N}}_j \subset \Omega_j$. For this particular choice, $\tilde{\mathcal{N}}_j = \{i = N_{j-1}+1, \ldots, N_j\} \bigcup \{i : \phi_i^{(j)} \neq \phi_i^{(j-1)},\ i = 1, \ldots, N_{j-1}\}$, the following result from [11] establishes a bound for the number of nodes used for smoothing (those created in Ω_j by the BEK procedure) so that the BPX preconditioner has provably optimal (linear) computational complexity per iteration.

Lemma 4.1. *The total number of nodes used for smoothing satisfies the bound:*

$$\sum_{j=0}^{J} \tilde{N}_j \leq \frac{5}{3} N_J - \frac{2}{3} N_0. \tag{4.1}$$

Proof. See [11, Lemma 1]. □

A similar result for 2D red-green refinement was given by Oswald [29, page 95]. In the general case of local smoothing operators which involve smoothing over newly created basis functions plus some additional set of local neighboring basis functions, one can extend the arguments from [11] and [29] using shape regularity.

5. Establishing optimality of the BPX norm equivalence

In this section, we extend the Dahmen-Kunoth framework to three spatial dimensions; the extension closely follows the original work. However, the general case for $d \geq 1$ spatial dimensions is not in the literature, and therefore we present it below.

For linear g, the element mass matrix gives rise to the following useful formula.

$$\|g\|^2_{L_2(\tau)} = \frac{\text{volume}(\tau)}{(d+1)(d+2)} \left(\sum_{i=1}^{d+1} g(x_i)^2 + [\sum_{i=1}^{d+1} g(x_i)]^2\right), \tag{5.1}$$

where, $i = 1, \ldots, d+1$ and x_i is a vertex of τ, $d = 2, 3$. In view of (5.1), we have that

$$\|\hat{\phi}_i^{(j)}\|^2_{L_2(\Omega)} = 2^{dL_{j,i}} \frac{\text{volume}(\text{supp } \hat{\phi}_i^{(j)})}{(d+1)(d+2)}.$$

Since the min in (3.7) is attained, there exists at least one $\tau \in \text{supp } \hat{\phi}_i^{(j)}$ such that $L(\tau) = L_{j,i}$. By (3.2) we have

$$2^{L_{j,i}} \approx \frac{\text{diam}(\tau)}{\text{diam}(\bar{\tau})}. \tag{5.2}$$

Also,

$$\text{volume}(\text{supp } \hat{\phi}_i^{(j)}) \approx \sum_{i=1}^{E} \text{diam}^d(\tau_i), \;\; \tau_i \in \text{supp } \hat{\phi}_i^{(j)}. \tag{5.3}$$

By (3.5), we have

$$\text{diam}(\tau_i) \approx \text{diam}(\tau). \tag{5.4}$$

Combining (5.3) and (5.4), we conclude

$$\text{volume}(\text{supp } \hat{\phi}_i^{(j)}) \approx E \, \text{diam}^d(\tau). \tag{5.5}$$

Finally then, (5.2) and (5.5) yield

$$2^{dL_{j,i}} \text{volume}(\text{supp } \hat{\phi}_i^{(j)}) \approx E \, \frac{1}{\text{diam}^d(\bar{\tau})}.$$

E is a uniformly bounded constant by shape regularity. One can view the size of any tetrahedron in $\mathcal{T}_0$, in particular size of $\bar{\tau}$, as a constant. The reason is the following: A.3.4 and A.3.5 force every tetrahedron at any $\mathcal{T}_j$ to be geometrically similar to some tetrahedron in $\mathcal{T}_0$ or to a tetrahedron arising from an irregular refinement of some tetrahedron in $\mathcal{T}_0$, hence, to some tetrahedron of a fixed finite collection. Combining the two arguments above, we have established that

$$\|\hat{\phi}_i^{(j)}\|_{L_2(\Omega)} \eqsim 1, \ x_i \in \mathcal{N}_j. \tag{5.6}$$

Let $g = \sum_{x_i \in \mathcal{N}_j} \hat{u}_i \hat{\phi}_i^{(j)} \in \mathcal{S}_j$. For any $\tau \in \mathcal{T}_j$ we have that

$$\|g\|_{L_2(\tau)}^2 \leq c \sum_{x_i \in \mathcal{N}_{j,\tau}} |\hat{u}_i|^2 \|\hat{\phi}_i^{(j)}\|_{L_2(\Omega)}^2, \tag{5.7}$$

where $\mathcal{N}_{j,\tau} = \{x_i \in \mathcal{N}_j : x_i \in \tau\}$, which is uniformly bounded in $\tau \in \mathcal{T}_j$ and $j \in \mathbf{b}N_0$. By the scaling (3.9), we get equality in the estimate below. The inequality is a standard inverse inequality where one bounds $g(x_i)$ using formula (5.1) and by handling the volume in the formula by (3.2):

$$|\hat{u}_i|^2 = 2^{-dL_{j,i}} |g(x_i)|^2 \leq c\, 2^{-dL_{j,i}} 2^{dL_{j,i}} \|g\|_{L_2(\tau)}^2. \tag{5.8}$$

Now, we are ready to establish that our basis is an L_2-stable Riesz basis as in (3.8). This is achieved by simply summing up over $\tau \in \mathcal{T}_j$ in (5.7) and (5.8) and using (5.6). L_2 stability in the Riesz sense allows us to establish the Bernstein inequality (2.5).

Lemma 5.1. *For the scaled basis (3.9), the Bernstein inequality (2.5) holds for* $\beta = 3/2$

Proof. (5.6) with (5.7) and (5.8) assert that the scaled basis (3.9) is stable in the sense of (3.8). Hence, (2.5) holds by [29, Theorem 4]. Note that the proof actually works independently of the spatial dimension. □

6. Lower bound in the norm equivalence

The Jackson inequality for Besov spaces is defined as follows:

$$\inf_{g \in \mathcal{S}_J} \|f - g\|_{L_p} \leq c\, \omega_\alpha(f, 2^{-J})_p, \ f \in L_p(\Omega), \tag{6.1}$$

where c is a constant independent of f and J, and α is an integer. In the uniform refinement setting, (6.1) is used to obtain the lower bound in (2.7). However, in the local refinement setting, (6.1) holds only for functions whose singularities are somehow well-captured by the mesh geometry. For instance, if a mesh is designed to pick up the singularity at $x = 0$ of $y = 1/x$, then on the same mesh we will not be able to recover a singularity at $x = 1$ of $y = 1/(x-1)$. Hence the Jackson inequality (6.1) cannot hold in a general setting, i.e. for $f \in W_p^k$. In order to get the lower bound in (2.7), we focus on estimating θ_J directly, as in [19] for the 2D setting.

To begin we borrow the quasi-interpolant construction from [19], extending it to the three-dimensional setting. Let $\tau \in \mathcal{T}_j$ be a tetrahedron with vertices x_1, x_2, x_3, x_4. Clearly the restrictions of $\hat{\phi}_i^{(j)}$ to τ are linearly independent over τ where $x_i \in \{x_1, x_2, x_3, x_4\}$. Then, there exists a unique set of linear polynomials $\psi_1^\tau, \psi_2^\tau, \psi_3^\tau, \psi_4^\tau$ such that

$$\int_\tau \hat{\phi}_k^{(j)}(x,y,z) \psi_l^\tau(x,y,z)\, dxdydz = \delta_{kl}, \ x_k, x_l \in \{x_1, x_2, x_3, x_4\}. \tag{6.2}$$

For $x_i \in \mathcal{N}_j$ and $\tau \in \mathcal{T}_j$, define a function for $x_i \in \tau$

$$M_i^{(j)}(x,y,z) = \begin{cases} \frac{1}{E_i} \psi_i^\tau(x,y,z), & (x,y,z) \in \tau \\ 0, & (x,y,z) \notin \operatorname{supp} \hat{\phi}_i^{(j)} \end{cases}, \tag{6.3}$$

where E_i is the number of tetrahedra in $\mathcal{T}_j$ in supp $\hat{\phi}_i^{(j)}$. By (6.2) and (6.3), we obtain

$$(M_k^{(j)}, \hat{\phi}_l^{(j)}) = \int_\Omega M_k^{(j)}(x,y,z)\hat{\phi}_l(x,y,z)\,dxdydz = \delta_{kl},\ x_k, x_l \in \mathcal{N}_j. \tag{6.4}$$

We can now define a quasi-interpolant, in fact a *projection* onto $\mathcal{S}_j$, such that

$$(\tilde{Q}_j f)(x,y,z) = \sum_{x_i \in \mathcal{N}_j} (f, M_i^{(j)})\hat{\phi}_i^{(j)}(x,y,z). \tag{6.5}$$

As remarked earlier, due to (6.3) the slice operator term $\tilde{Q}_j - \tilde{Q}_{j-1}$ will vanish outside the refined set Ω_j defined in (2.1). One can easily observe by (5.6) and (6.4) that

$$\|M_i^{(j)}\|_{L_2(\Omega)} \approx 1,\ x_i \in \mathcal{N}_j,\ j \in \mathbf{b}N_0. \tag{6.6}$$

Letting $\Omega_{j,\tau} = \bigcup\{\tau' \in \mathcal{T}_j :\ \tau \cap \tau' \neq \emptyset\}$, we can conclude from (5.6) and (6.6) that

$$\|\tilde{Q}_j f\|_{L_2(\tau)} = \|\sum_{x_k \in \mathcal{N}_{j,\tau}} (f, M_l^{(j)})\hat{\phi}_k^{(j)}\|_{L_2(\tau)} \leq c\|f\|_{L_2(\Omega_{j,\tau})}. \tag{6.7}$$

We define now a subset of the triangulation where the refinement activity stops, meaning that all tetrahedra in $\mathcal{T}_j^*,\ j \leq m$ also belong to $\mathcal{T}_m$:

$$\mathcal{T}_j^* = \{\tau \in \mathcal{T}_j :\ L(\tau) < j,\ \Omega_{j,\tau} \cap \tau' = \emptyset,\ \forall \tau' \in \mathcal{T}_j \text{ with } L(\tau') = j\}. \tag{6.8}$$

Due to the local support of the dual basis functions $M_i^{(j)}$ and the fact that $\tilde{Q}_j$ is a projection, one gets for $g \in \mathcal{S}_J$:

$$\|g - \tilde{Q}_j g\|_{L_2(\tau)} = 0,\ \tau \in \mathcal{T}_j^*. \tag{6.9}$$

Since $\tilde{Q}_j$ is a projection onto linear finite element space, it fixes polynomials of degree at most 1 (i.e. $\Pi_1(\Re^3)$). Using this fact and (6.7), we arrive:

$$\begin{aligned} \|g - \tilde{Q}_j g\|_{L_2(\tau)} &\leq \|g - P\|_{L_2(\tau)} + \|\tilde{Q}_j(P - g)\|_{L_2(\tau)} \\ &\leq c\,\|g - P\|_{L_2(\Omega_{j,\tau})}, \quad \tau \in \mathcal{T}_j \setminus \mathcal{T}_j^*. \end{aligned} \tag{6.10}$$

We would like to bound the right hand side of (6.10) in terms of a modulus of smoothness in order to reach a Jackson-type inequality. Following [19], we utilize a modified modulus of smoothness for $f \in L_p(\Omega)$

$$\tilde{\omega}_k(f,t,\Omega)_p^p = t^{-s}\int_{[-t,t]^s} \|\Delta_h^k f\|_{L_p(\Omega_{k,h})}^p\,dh.$$

They can be shown to be equivalent:

$$\tilde{\omega}_{k+1}(f,t,\Omega)_p \approx \omega_{k+1}(f,t,\Omega)_p.$$

The equivalence in the one-dimensional setting can be found in [20, Lemma 5.1].

For τ a simplex in $\Re^d$ and $t = \operatorname{diam}(\tau)$, a Whitney estimate shows that [21, 28, 33]

$$\inf_{P \in \Pi_k(\Re^d)} \|f - P\|_{L_p(\tau)} \leq c\tilde{\omega}_{k+1}(f,t,\tau)_p, \tag{6.11}$$

where c depends only on the smallest angle of τ but not on f and t. The reason why $\tilde{Q}_j$ works well for tetrahedralization in 3D is the fact that the Whitney estimate (6.11) remains valid for any spatial dimension. $\mathcal{T}_j \setminus \mathcal{T}_j^*$ is the part of the tetrahedralization $\mathcal{T}_j$ where refinement is active at every level. Then, in view of (3.5)

$$\operatorname{diam}(\Omega_{j,\tau}) \approx 2^{-j},\ \tau \in \mathcal{T}_j \setminus \mathcal{T}_j^*.$$

Taking the inf over $P \in \Pi_1(\Re^3)$ in (6.10) and using the Whitney estimate (6.11) we conclude

$$\|g - \tilde{Q}_j g\|_{L_2(\tau)} \le c\tilde{\omega}_2(g, 2^{-j}, \Omega_{j,\tau})_2.$$

Recalling (6.9) and summing over $\tau \in \mathcal{T}_j \setminus \mathcal{T}_j^*$ gives rise to

$$\|g - \tilde{Q}_j g\|_{L_2(\Omega)} \le c\tilde{\omega}_2(g, 2^{-j}, \Omega)_2 \le \tilde{c}\,\omega_2(g, 2^{-j}, \Omega)_2,$$

where we have switched from the modified modulus of smoothness to the standard one. Since Q_j is an orthogonal projection, we have the following:

$$\|g - Q_j g\| \le \|g - \tilde{Q}_j g\|.$$

Using the above inequality with (2.6) one then has

$$v_J = O(1),\ J \to \infty. \tag{6.12}$$

7. The WHB preconditioner

In local refinement, HB methods enjoy an optimal complexity of $O(N_j - N_{j-1})$ per iteration per level (resulting in $O(N_J)$ overall complexity per iteration) by only using degrees of freedom (DOF) corresponding to $\mathcal{S}_j^f$. However, HB methods suffer from suboptimal iteration counts or equivalently suboptimal condition number. The BPX decomposition $\mathcal{S}_j = \mathcal{S}_{j-1} \oplus (Q_j - Q_{j-1})\mathcal{S}_j$ gives rise to basis functions which are not locally supported, but they decay rapidly outside a local support region. This allows for locally supported approximations, and in addition the WHB methods [34, 35, 36] can be viewed as an approximation of the wavelet basis stemming from the BPX decomposition [23]. A similar wavelet-like multilevel decomposition approach was taken in [32], where the orthogonal decomposition is formed by a discrete L_2-equivalent inner product. This approach utilizes the same BPX two-level decomposition [31, 32]. The WHB preconditioner is defined as follows:

$$Hu := \sum_{j=0}^{J} 2^{j(d-2)} \sum_{i \in \mathcal{N}_j^f} (u, \psi_i^{(j)})\psi_i^{(j)}, \tag{7.1}$$

where $\psi_i^{(j)} = (\tilde{Q}_j - \tilde{Q}_{j-1})\phi_i^{(j)}$. The WHB preconditioner uses the modified basis (where as the BPX preconditioner uses the standard nodal basis) where the projection operator used is defined as in (7.5). In the WHB setting, these operators are chosen to satisfy the following three properties [5]:

$$\tilde{Q}_j\,|_{\mathcal{S}_j} = I, \tag{7.2}$$

$$\tilde{Q}_j\tilde{Q}_k = \tilde{Q}_{\min\{j,k\}}, \tag{7.3}$$

$$\|(\tilde{Q}_j - \tilde{Q}_{j-1})u^{(j)}\|_{L_2} \eqsim \|u^{(j)}\|_{L_2},\ u^{(j)} \in (I_j - I_{j-1})\mathcal{S}_j. \tag{7.4}$$

As indicated in (2.2), the WHB smoother acts on only the fine DOF, i.e. $\mathcal{N}_j^f$, and hence is an approximation to fine-fine discretization operator; $A_{ff}^{(j)} : \mathcal{S}_j^f \to \mathcal{S}_j^f$, where $\mathcal{S}_j^f := (\tilde{Q}_j - \tilde{Q}_{j-1})\mathcal{S}_j$ and f stands for fine. On the other hand, the BPX smoother acts on a slightly bigger set than fine DOF, $\mathcal{N}_j^f \subseteq \tilde{\mathcal{N}}_j$ typically, union of fine DOF and their corresponding coarse fathers.

The WHB preconditioner introduced in [34, 35] is, in some sense, the best of both worlds. While the condition number of the HB preconditioner is stabilized by inserting Q_j in the definition of $\tilde{Q}_j$, somehow employing the operators $I_j - I_{j-1}$ at the same time guarantees optimal computational and storage cost per iteration. The operators which will be seen to meet both goals at the same time are:

$$\tilde{Q}_k = \prod_{j=k}^{J-1} I_j + Q_j^a(I_{j+1} - I_j), \tag{7.5}$$

with $\tilde{Q}_J = I$. The exact L_2-projection Q_j is replaced by a computationally feasible approximation $Q_j^a : L_2 \to \mathcal{S}_j$. To control the approximation quality of Q_j^a, a small fixed tolerance γ is introduced:

$$\|(Q_j^a - Q_j)u\|_{L_2} \leq \gamma \|Q_j u\|_{L_2}, \ \forall u \in L_2(\Omega). \tag{7.6}$$

In the limiting case $\gamma = 0$, $\tilde{Q}_k$ reduces to the exact L_2-projection on $\mathcal{S}_J$ by (7.2):

$$\tilde{Q}_k = Q_k \ I_{k+1} Q_{k+1} \dots I_{J-1} Q_{J-1} \ I_J = Q_k Q_{k+1} \dots Q_{J-1} = Q_k.$$

Following [34, 35], the properties (7.2), (7.3), and (7.4) can be verified for $\tilde{Q}_k$ as follows:

• Property (7.2): Let $u^{(k)} \in \mathcal{S}_k$. Since $(I_{j+1} - I_j)u^{(k)} = 0$ and $I_j u^{(k)} = u^{(k)}$ for $k \leq j$, then $[I_j + Q_j^a(I_{j+1} - I_j)](u^{(k)}) = u^{(k)}$, verifying (7.2) for $\tilde{Q}_k$. It also implies

$$\tilde{Q}_k^2 = \tilde{Q}_k. \tag{7.7}$$

• Property (7.3): Let $k \leq l$, then by (7.7)

$$\tilde{Q}_k \tilde{Q}_l = [(I_k + Q_k^a(I_{k+1} - I_k)) \dots (I_{l-1} + Q_{l-1}^a(I_l - I_{l-1})) \ \tilde{Q}_l] \tilde{Q}_l = \tilde{Q}_k. \tag{7.8}$$

Since $\tilde{Q}_k u \in \mathcal{S}_k$ and $\mathcal{S}_k \subset \mathcal{S}_l$, then by (7.2) we have

$$\tilde{Q}_l(\tilde{Q}_k u) = \tilde{Q}_k u. \tag{7.9}$$

Finally, (7.3) then follows from (7.8) and (7.9).

• Property (7.4): This is an implication of Lemma 7.1.

For an overview, we list the corresponding slice spaces for the preconditioners of interest:

$$\begin{array}{llll} \text{HB:} & \mathcal{S}_j^f & = & (I_j - I_{j-1})\mathcal{S}_j, \\ \text{BPX:} & \mathcal{S}_j^f & = & (Q_j - Q_{j-1})\mathcal{S}_j, \\ \text{WHB:} & \mathcal{S}_j^f & = & (\tilde{Q}_j - \tilde{Q}_{j-1})\mathcal{S}_j = (I - Q_{j-1}^a)(I_j - I_{j-1})\mathcal{S}_j, \ \tilde{Q}_j \text{ as in (7.5).} \end{array}$$

The WHB smoother only acts on the fine DOF. Then, in the generic multilevel preconditioner notation, the WHB preconditioner can be written in the following form:

$$Bu := \sum_{j=0}^{J} B_{ff}^{(j)^{-1}} (\tilde{Q}_j - \tilde{Q}_{j-1})u. \tag{7.10}$$

B_{ff} is chosen to be a spectrally equivalent operator to fine-fine discretization operator $A_{ff}^{(j)}$. Since the smoother and property (7.4) both rely on a well-conditioned $A_{ff}^{(j)}$, we discuss this next.

7.1. **Well-conditioned $A_{ff}^{(j)}$.** The lemma below is essential to extend the existing results for quasiuniform meshes [34, Lemma 6.1] or [35, Lemma 2] to the locally refined ones. $\mathcal{S}_j^{(f)} = (I_j - I_{j-1})\mathcal{S}_j$ denotes the HB slice space.

Lemma 7.1. *Let $\mathcal{T}_j$ be constructed by the local refinements under consideration. Let $\mathcal{S}_j^f = (I - \tilde{Q}_{j-1})\mathcal{S}_j^{(f)}$ be the modified hierarchical subspace where $\tilde{Q}_{j-1}$ is any L_2-bounded operator. Then, there are constants c_1 and c_2 independent of j such that*

$$c_1 \|\phi^f\|_X^2 \leq \|\psi^f\|_X^2 \leq c_2 \|\phi^f\|_X^2, \ X = H^1, L_2, \tag{7.11}$$

holds for any $\psi^f = (I - \tilde{Q}_{j-1})\phi^f \in \mathcal{S}_j^f$ *with* $\phi^f \in \mathcal{S}_j^{(f)}$.

Proof. The Cauchy-Schwarz like inequality [8] is central to the proof: There exists $\delta \in (0,1)$ independent of the mesh size or level j such that

$$(1-\delta^2)(\nabla\phi^f, \nabla\phi^f) \leq (\nabla(\phi^c+\phi^f), \nabla(\phi^c+\phi^f)),\ \forall \phi^c \in \mathcal{S}_{j-1}, \phi^f \in \mathcal{S}_j^{(f)}. \quad (7.12)$$
$$(1-\delta^2)\|\phi^f\|_{L_2}^2 \leq c|\phi^c+\phi^f|_{H^1}^2 \text{ (by Poincare inequality and (7.12))}. \quad (7.13)$$

Combining (7.12) and (7.13): $(1-\delta^2)\|\phi^f\|_{H^1}^2 \leq \|\phi^c+\phi^f\|_{H^1}^2$. Choosing $\phi^c = -\tilde{Q}_{j-1}\phi^f$, we get the lower bound: $(1-\delta^2)\|\phi^f\|_{H^1}^2 \leq \|\psi^f\|_{H^1}^2$.

Let Ω_j^f denote the support of basis functions corresponding to $\mathcal{N}_j^f$. Due to nested refinement, triangulation on Ω_j^f is quasiuniform. One can analogously introduce a triangulation hierarchy where all the simplices are exposed to uniform refinement: $\mathcal{T}_j^f := \{\tau \in \mathcal{T}_j : L(\tau) = j\} = \mathcal{T}_j|_{\Omega_j^f}$. Hence, $\mathcal{T}_j^f$ becomes a quasiuniform tetrahedralization and the inverse inequality holds for $\mathcal{S}_j^f$. To derive the upper bound: The right scaling is obtained by father-son size relation, and by the inverse inequalities and L_2-boundedness of $\tilde{Q}_{j-1}$, one gets

$$\|\psi^f\|_{H^1}^2 \leq c_0 2^{2j}\|\psi^f\|_{L_2}^2 \leq c_0 2^{2j}\left(1+\|\tilde{Q}_{j-1}\|_{L_2}\right)^2\|\phi^f\|_{L_2}^2 \leq c2^{2j}\|\phi^f\|_{L_2}^2.$$

The slice space $\mathcal{S}_j^{(f)}$ is oscillatory. Then there exists c such that $\|\phi^f\|_{L_2}^2 \leq c2^{-2j}\|\phi^f\|_{H^1}^2$. Hence, $\|\psi^f\|_{H^1}^2 \leq c\|\phi^f\|_{H^1}^2$. The case for $X = L_2$ can be established similarly. □

Using the above tools, one can establish that $A_{ff}^{(j)}$ is well-conditioned. Namely,

$$c_1 2^{2j} \leq \lambda_{j,\min}^f \leq \lambda_{j,\max}^f \leq c_2 2^{2j}, \quad (7.14)$$

where $\lambda_{j,\min}^f$ and $\lambda_{j,\max}^f$ are the smallest and largest eigenvalues of $A_{ff}^{(j)}$, and c_1 are and c_2 both independent of j. For details see [34, Lemma 4.3] or [35, Lemma 3].

8. THE FUNDAMENTAL ASSUMPTION AND WHB OPTIMALITY

As in the BPX splitting, the main ingredient in the WHB splitting is the L_2-projection. Hence, the stability of the BPX splitting is still important in the WHB splitting. The lower bound in the BPX norm equivalence is the *fundamental assumption* for the WHB preconditioner. Utilizing a local projection $\tilde{Q}_j$, BPX lower bound was verified earlier for 3D local red-green (BEK) refinement procedure. The same result easily holds for the projection Q_j. Dahmen and Kunoth [19] verified BPX lower bound for the 2D red-green refinement procedures.

Before getting to the stability result we remark that the existing perturbation analysis of WHB is one of the primary insights in [34, 35]. Although not observed in [34, 35], the result does not require substantial modification for locally refined meshes. Let $e_j := (\tilde{Q}_j - Q_j)u$ be the error, then the following holds.

Lemma 8.1. *Let* γ *be as in (7.6). There exists an absolute* c *satisfying:*

$$\sum_{j=0}^{J} 2^{2j}\|e_j\|_{L_2}^2 \leq c\gamma^2 \sum_{j=0}^{J} 2^{2j}\|(Q_j - Q_{j-1})u\|_{L_2}^2,\ \forall u \in \mathcal{S}_J. \quad (8.1)$$

Proof. [34, Lemma 5.1] or [35, Lemma 1]. □

We arrive now at the primary result, which indicates that the WHB slice norm is optimal on the class of locally refined meshes under consideration.

Theorem 8.2. *If there exists sufficiently small* γ_0 *such that (7.6) is satisfied for* $\gamma \in [0, \gamma_0)$*, then*

$$\|u\|^2_{\text{WHB}} = \sum_{j=0}^{J} 2^{2j} \|(\tilde{Q}_j - \tilde{Q}_{j-1})u\|^2_{L_2} \eqsim \|u\|^2_{H^1}, \; u \in \mathcal{S}_J. \tag{8.2}$$

Proof. Observe that

$$\begin{aligned} (\tilde{Q}_j - \tilde{Q}_{j-1})u &= (\tilde{Q}_j - Q_j)u - (\tilde{Q}_{j-1} - Q_{j-1})u + (Q_j - Q_{j-1})u \\ &= e_j - e_{j-1} + (Q_j - Q_{j-1})u. \end{aligned} \tag{8.3}$$

This gives

$$\begin{aligned} \sum_{j=0}^{J} 2^{2j} \|(\tilde{Q}_j - \tilde{Q}_{j-1})u\|^2_{L_2} &\le c \sum_{j=0}^{J} 2^{2j} \|(Q_j - Q_{j-1})u\|^2_{L_2} + c \sum_{j=0}^{J} 2^{2j} \|e_j\|^2_{L_2} \\ &\le c(1+\gamma^2) \sum_{j=0}^{J} 2^{2j} \|(Q_j - Q_{j-1})u\|^2_{L_2} \text{ (using (8.1))} \\ &\le c\|u\|^2_{H^1}. \end{aligned}$$

Let us now proceed with the upper bound. The Bernstein inequality (2.5) holds for $\mathcal{S}_j$ [1, 19] for the local refinement procedures. Hence we are going to utilize an inequality involving the Besov norm $\|\cdot\|_{B^1_{2,2}}$ which naturally fits our framework when the moduli of smoothness is considered in (2.5). The following important inequality holds, provided that (2.5) holds [29, page 39]:

$$\|u\|^2_{B^1_{2,2}} \le c \sum_{j=0}^{J} 2^{2j} \|u^{(j)}\|^2_{L_2}, \tag{8.4}$$

for any decomposition such that $u = \sum_{j=0}^{J} u^{(j)}, \; u^{(j)} \in \mathcal{S}_j$, in particular for $u^{(j)} = (\tilde{Q}_j - \tilde{Q}_{j-1})u$. Then the upper bound holds due to $H^1(\Omega) \cong B^1_{2,2}(\Omega)$. ☐

Remark 8.3. *The following equivalence is used for the upper bound in the proof of Theorem 8.2 on uniformly refined meshes* [35, Lemma 4].

$$c_1\|u\|^2_{H^1} \le \inf_{u=\sum_{j=0}^{J} u^{(j)}, \; u^{(j)} \in \mathcal{S}_j} \sum_{j=0}^{J} 2^{2j} \|u^{(j)}\|^2_{L_2} \le c_2 \|u\|^2_{H^1}.$$

Let us emphasize that the left hand side holds in the presence of the Bernstein inequality (2.5), and the right hand side holds in the simultaneous presence of Bernstein and Jackson inequalities. However, the Jackson inequality cannot hold under local refinement procedures (cf. counter example in section 6). That is why we can utilize only the left hand side of the above equivalence as in (8.4).

Now, we have all the required estimates at our disposal to establish the optimality of WHB preconditioner for 2D/3D red-green refinement procedures for $p \in L_\infty(\Omega)$. We would like to emphasize that our framework supports any spatial dimension $d \ge 1$, provided that the necessary geometrical abstractions are in place.

Theorem 8.4. *If BPX lower bound holds and if there exists sufficiently small γ_0 such that (7.6) is satisfied for $\gamma \in (0, \gamma_0)$, then for B in (7.10):*

$$(Bu, u) \eqsim \|u\|^2_{H^1}.$$

Proof. $B^{(j)}_{ff}$ is spectrally equivalent to $A^{(j)}_{ff}$. Since $A^{(j)}_{ff}$ is a well-conditioned matrix, using (7.14) it is spectrally equivalent to $2^{2j}I$. The result follows from Theorem 8.2. □

An extension to multiplicative WHB preconditioner is also possible under additional assumptions. These results will not be reported here.

9. H^1-STABLE L_2-PROJECTION

The involvement of $\tilde{Q}_j$ in the multilevel decomposition makes it the most crucial element in the stabilization. We then come to the central question: Which choice of $\tilde{Q}_j$ can provide an optimal preconditioner? The following theorem sets a guideline for picking $\tilde{Q}_j$. It shows that H^1-stability of the $\tilde{Q}_j$ is actually a *necessary condition* for obtaining an optimal preconditioner.

Theorem 9.1. [34, 35]. *If $\tilde{Q}_j$ induces an optimal preconditioner, namely for $u \in \mathcal{S}_J$, $\sum_{j=0}^J 2^{2j}\|(\tilde{Q}_j - \tilde{Q}_{j-1})u\|^2_{L_2} \eqsim \|u\|^2_{H^1}$, then there exists an absolute constant c such that*

$$\|\tilde{Q}_k u\|_{H^1} \le c\, \|u\|_{H^1},\ \forall k \le J.$$

Proof. Using the multilevel decomposition and (7.3), we get:
$\tilde{Q}_k u = \sum_{j=0}^k (\tilde{Q}_j - \tilde{Q}_{j-1})u$. Since $\tilde{Q}_j$ induces an optimal preconditioner, there exist two absolute constants σ_1 and σ_2:

$$\sigma_1\|u\|^2_{H^1} \le \sum_{j=0}^J 2^{2j}\|(\tilde{Q}_j - \tilde{Q}_{j-1})u\|^2_{L_2} \le \sigma_2\|u\|^2_{H^1},\ \forall u \in \mathcal{S}_J. \tag{9.1}$$

Using (9.1) for $\tilde{Q}_k u$:

$$\|\tilde{Q}_k u\|^2_{H^1} \le \frac{1}{\sigma_1}\sum_{j=0}^k 2^{2j}\|(\tilde{Q}_j - \tilde{Q}_{j-1})u\|^2_{L_2} \le \frac{1}{\sigma_1}\sum_{j=0}^J 2^{2j}\|(\tilde{Q}_j - \tilde{Q}_{j-1})u\|^2_{L_2} \le \frac{\sigma_2}{\sigma_1}\|u\|^2_{H^1}.$$

□

As a consequence of Theorem 9.1 we have

Corollary 9.2. *L_2-projection restricted to $\mathcal{S}_j$, $Q_j|_{\mathcal{S}_j} : L_2 \to \mathcal{S}_j$, is H^1-stable on 2D and 3D locally refined meshes by red-green refinement procedures.*

Proof. Optimality of the BPX norm equivalence on the above locally refined meshes was already established. Application of Theorem 9.1 with Q_j proves the result. Alternatively, the same result can be obtained through Theorem 9.1 applied to the WHB framework. Theorem 8.2 will establish the optimality of the WHB preconditioner for the local refinement procedures. Hence, the operator $\tilde{Q}_j$ restricted to $\mathcal{S}_j$ is H^1-stable. Since $\tilde{Q}_j$ is none other than Q_j in the limiting case, we can also conclude the H^1-stability of the L_2-projection. □

Our stability result appears to be the first *a priori* H^1-stability for the L_2-projection on these classes of locally refined meshes. H^1-stability of L_2-projection is guaranteed for the subset $\mathcal{S}_j$ of $L_2(\Omega)$, not for all of $L_2(\Omega)$. This question is currently undergoing intensive study in the finite element and approximation theory community. The existing theoretical results, mainly in [15, 18], involve *a posteriori* verification of somewhat complicated mesh conditions after refinement has taken place. If such mesh conditions are not satisfied, one has to redefine the mesh. The mesh conditions mentioned require that the simplex sizes do not change drastically between regions of refinement. In this context, quasiuniformity in the support of a basis function becomes crucial. This type of local quasiuniformity is usually called as *patchwise quasiuniformity*. Local quasiuniformity requires neighbor generation relations as in (3.1), neighbor size relations, and shape regularity of the mesh. It was shown in [1] that patchwise quasiuniformity holds also for 3D marked tetrahedron bisection [24] and for 2D newest vertex bisection [25, 30]. These are then promising refinement procedures for which H^1-stability of the L_2-projection can be established.

10. Conclusion

In this article, we examined the Bramble-Pasciak-Xu (BPX) norm equivalence in the setting of local 3D mesh refinement. In particular, we extended the 2D optimality result for BPX due to Dahmen and Kunoth to the local 3D red-green refinement procedure introduced by Bornemann-Erdmann-Kornhuber (BEK). The extension involved establishing that the locally enriched finite element subspaces produced by the BEK procedure allow for the construction of a scaled basis which is formally Riesz stable. This in turn rested entirely on establishing a number of geometrical relationships between neighboring simplices produced by the local refinement algorithms. We remark again that shape regularity of the elements produced by the refinement procedure is insufficient to construct a stable Riesz basis for finite element spaces on locally adapted meshes. The d-vertex adjacency generation bound for simplices in $\Re^d$ is the primary result required to establish patchwise quasiuniformity for stable Riesz basis construction, and this result depends delicately on the particular details of the local refinement procedure rather than on shape regularity of the elements. We also noted in §3 that these geometrical properties have been established in [1] for purely bisection-based refinement procedures that have been shown to be asymptotically non-degenerate, and therefore also allow for the construction of a stable Riesz basis.

We also examined the wavelet modified hierarchical basis (WHB) methods of Vassilevski and Wang, and extended their original quasiuniformity-based framework and results to local 2D and 3D red-green refinement scenarios. A critical step in the extension involved establishing the optimality of the BPX norm equivalence for the local refinement procedures under consideration, as established in the first part of this article. With the local refinement extension of the WHB analysis framework presented here, we established the optimality of the WHB preconditioner on locally refined meshes in both 2D and 3D under the minimal regularity assumptions required for well-posedness. An interesting implication of the optimality of WHB preconditioner was the *a priori* H^1-stability of the L_2-projection. Existing *a posteriori* approaches in the literature dictate a reconstruction of the mesh if such conditions cannot be satisfied.

The theoretical framework established here supports arbitrary spatial dimension $d \geq 1$, and therefore allows extension of the optimality results, the H^1-stability of L_2-projection results, and the various supporting results to arbitrary $d \geq 1$. We indicated clearly which geometrical properties must be re-established to show BPX optimality for spatial dimension $d \geq 4$. All of the results here require no smoothness assumptions on the PDE coefficients beyond those required for well-posedness in H^1.

To address the practical computational complexity of implementable versions of the BPX and WHB preconditioners, we indicated how the number of degrees of freedom used for the smoothing step can be shown to be bounded by a constant times the number of degrees of freedom introduced at that level of refinement. This indicates that practical implementable versions of the BPX and WHB preconditioners for the local 3D refinement setting considered here have provably optimal (linear) computational complexity per iteration. A detailed analysis of both the storage and per-iteration computational complexity questions arising with BPX and WHB implementations can be found in the second article [2].

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E-mail address: burak@cct.lsu.edu

DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LA 70803, USA

CENTER FOR COMPUTATION AND TECHNOLOGY, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LA 70803, USA

E-mail address: mholst@math.ucsd.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT SAN DIEGO, LA JOLLA, CA 92093, USA